1962 Ph.D. Qualifying

Algebra

40% 1. Give a resume of the main notions and theorems in two of the following subjects. Try to be concise and well-organized.

(a) Structure of finite fields.

(b) Sylow theorems.

(c) The group algebra of a finite group.

(d) The Galois theory of solvability of equations by radicals.

(e) Jordan canonical form for matrices over the complex numbers.

(f) Ideal theory in rings of algebraic integers in finite extensions of the rationals.

(g) Direct decomposition theorems for finite and infinite Abelian groups.

In each of the following problems 2 - 4 choose one part (a) or (b). Each such part is worth 20%. Some of these parts may also have two sub-parts (i), (ii), both of which must be worked for the choice of that part.

2. (a) An arbitrary (possibly infinite) group is called solvable if it has a finite maximal normal series with Abelian quotients. Show that any homomorphic image of a solvable group is again solvable.

(b) (Two parts)

(i) If G is a transitive permutation group on a set S, and if $a, b \in S$ what is the relation between the subgroup G_a that leaves a fixed and the subgroup G_b that leaves b fixed? Prove your statement.

(ii) If G is a group of permutations on a set S of $p^r + 1$ elements and order$(G) = p^n$, can G be transitive? Prove your statement.

$p = 2, r = 1$

3. (a) Suppose K is a field, $f(x)$ is a polynomial over K, and L is a root field (splitting field) of $f(x)$ over K. Suppose $g(x)$ is any irreducible polynomial over K, and that $g(\alpha) = 0$ for some $\alpha \in L$. Prove that $g(x)$ has all its roots in L.

(b) (Two parts)

(i) Show that any complex number α which is a root of a polynomial with algebraic coefficients is necessarily algebraic (i.e., satisfies a polynomial with rational coefficients).

(ii) Find a specific polynomial with rational coefficients of which $\sqrt{2} + \sqrt[3]{3}$ is a root.

4. (a) (Two parts)

(i) Prove that any two bases for a finite dimensional vector space have the same number of elements.

(ii) Prove the same thing for any infinite-dimensional vector space.

(b) An inner product operation (x,y) on a vector space V is defined to be a symmetric, bilinear, positive definite function on $V \times V$ to the reals.

Prove: If V is a finite dimensional vector space over the reals with an inner product operation (x,y), and $f(x)$ is any linear functional on V, then there exists unique $y_1 \in V$ such that $f(x) = (x,y_1)$ for all x.