1963 Ph.D. Qualifying Examination

ALGEBRA

Problems 1 and 2 are required. Choose any three problems from problems 3-6. Specific instructions accompany each problem.

	Point Score
Problem	1 - 30%
	2 - 25%
	3 - 15%
	4 - 15%
	5 - 15%
	6 - 15%

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1. Let A be a commutative ring with unity; let X be an indeterminate over A. Some of the assertions below concerning A and A[X] are true; some are false. For each assertion, state whether it is true or false; for oneof the true assertions give a complete proof, and for one of the false assertions give a counter-example together with the necessary details.

   a) Any polynomial of degree $n > 0$ in A[X] has at most n roots in A.

   b) If A is Noetherian, so is A[X].

   c) If A is a principal ideal domain, so is A[X].

   d) If A has the descending chain condition, so does A[X].

   e) If A is a unique factorization domain, so is A[X].

   f) A[X] is a Euclidean Domain.

   g) Every prime ideal of A[X] is a maximal ideal.

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2. Do one of (a), (b), or (c).

a) Prove from first principles that a finite group whose order is divisible by the prime number p contains an element of order p.

b) (Two parts) A subgroup H of an arbitrary (not necessarily finite) group G is called a maximal subgroup of G if no subgroup of G (except G itself) properly contains H.

i) Show that every finitely generated group possesses a maximal subgroup.

ii) Is the statement in (i) valid without the hypothesis of finite generation? Proof or counter-example.

c) Let Q^* be the multiplicative group of all non-zero rational numbers. Let $\mathbb{Z}$ be the additive group of all integers. Prove that Q^* is isomorphic to a direct sum of a countable number of copies of $\mathbb{Z}$ and one copy of the cyclic group of order 2.

3. Do one of (a), (b), or (c).

a) Let k and K be fields, and suppose that K is a finite, separable extension of k, say of degree n. Give an upper bound for the number of subfields of K lying over k. (Your bound need not be tight; any finite upper bound will do.)

b) Let k be a field of characteristic $p > 0$. An element x of some extension field K of k is called purely inseparable over k if there exists a positive integer n such that

$$x^{p^n} \in k .$$

Prove that if x is both separable and purely inseparable over k it lies in k.

c) Determine explicitly the Galois Group of the polynomial

$$x^4 - 2$$

over the rational numbers.

4. Give a concise, well-organized resume of the main notions and theorems in one of the following subjects.

   a) Structure theory of rings with descending chain condition on left ideals.

   b) Direct decomposition theorems for finite and infinite Abelian groups.

   c) Ideal theoretic structure of the algebraic integers in finite extensions of the rationals.

   d) Representations of finite groups and the group algebra of a finite group.

5. Do either (a) or (b).

   a) Let V be a finite dimensional vector space over the field of real numbers. Let Q be a quadratic form on V. Prove: there exists a basis $b_1, \ldots, b_n$ of V such that

$$Q\left(\sum_{i=1}^{n} \alpha_i b_i\right) = \sum_{i=1}^{p} \alpha_i^2 - \sum_{i=p+1}^{s} \alpha_i^2 \qquad (\alpha_i \in \mathbb{R}) .$$

   Prove also that the numbers p, s above are independent of the choice of basis.

   b) Let k be a field, let X be an indeterminate over k, and let f be a polynomial of degree $n > 0$ in $k[X]$. Let V be a vector space of dimension n over k; let $e_0, \ldots, e_{n-1}$ be a basis for V over k. Suppose the polynomial f to have the form

$$f = a_0 + a_1 X + \dots + a_n X^n , \qquad a_n \neq 0 ,$$

and define a linear transformation $T : V \to V$ by

$$T(e_j) = e_{j+1} , \qquad j = 0 , \dots , n-2 ,$$
$$T(e_{n-1}) = \sum_{j=0}^{n-1} -\frac{a_j}{a_n} e_j .$$

A subspace W of V is called T-invariant if $T(W) \subseteq W$. Prove that there is a 1-1 correspondence between T-invariant subspaces of V and ideals of the residue class ring $k[X]/\mathfrak{U}$, where $\mathfrak{U}$ is the principal ideal generated by f. Show that f is irreducible over k if and only if V contains no non-trivial T-invariant subspaces.

6. Do one of (a), (b), or (c).

(a) Let A be a commutative ring with unity. Show that an element $x \in A$ is nilpotent (i.e., $x^n = 0$ for some $n > 0$) if and only if x belongs to every prime ideal of A.

(b) Let A be a Noetherian ring. Prove that every non-unit of A is a product of finitely many irreducible elements of A.

(c) Prove: An integral domain is a field if and only if it has the descending chain condition.