1965 Ph.D. Qualifying Examination

ALGEBRA

$\mathbb{C}$ = complexes

$\mathbb{Q}$ = rationals

$\mathbb{R}$ = reals

$\mathbb{Z}_n$ = integers mod n = cyclic group of order n .

I. Miscellaneous

1) Let h be the natural map of $\mathbb{Z}_{pq} \to \mathbb{Z}_p \oplus \mathbb{Z}_q$ given by $h(1) = (1,1)$. Assuming p and q relatively prime, how does one find r with $h(r) = (1,0)$?

2) $\mathbb{R}[x]/(x^3+x)$ is a direct sum of fields. What fields?

3) Give an example of a commutative ring with a prime ideal that is not maximal, and an example for the reverse situation.

4) Prove that any principal ideal domain satisfies the ascending chain condition.

5) Are the finite fields $GF(p^n)$ separable or inseparable over their prime fields? Are they perfect or imperfect?

6) Let p be prime; put $G = Z_p \oplus \dots \oplus Z_p$, n terms.

II. Algebra

1) a) Which of the following subfields of $\mathbb{C}$ go into themselves under all automorphisms of $\mathbb{C}$ [indicate reasons]: $\mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt[3]{2})$, $\mathbb{Q}(\pi)$, all algebraic numbers , $\mathbb{R}$.

b) D is an integral domain such that

α) the ascending chain condition holds, and

β) every proper not maximal ideal is product of two proper ideals.

Show: every proper ideal is product of maximal ideals.

2) a) Define the notion of primitive n-th root of unity (over $\mathbb{Q}$, say); show that they exist, and describe how one finds the cyclotomic polynomials φ_n (whose roots are precisely the primitive $\sqrt[n]{1}$'s).

b) For the polynomial x^5-2, determine the degree and the Galois group (at least a composition series) of $K/\mathbb{Q}$. What are the degree and Galois group of the subfield $\mathbb{Q}[\sqrt[5]{1}]$? Is x^5-2 irreducible over $\mathbb{Q}[\sqrt[5]{1}]$?

c) Let A be a Noetherian ring (commutative with 1), M a finitely generated A-module. Show: M has a composition series $M = M_1 \supset M_2 \supset \dots \supset M_n = (0)$ with all M_i/M_{i+1} of the form A/p, p prime ideal.

III. Group Theory

1) a). Prove that finite p-groups ($\neq \{e\}$) have nontrivial center.

b). Let G, H be finitely generated abelian groups. Suppose G is isomorphic to a subgroup of H, and conversely H to one G. Show: G and H are isomorphic.

2) a). G = finite group; H = a subgroup such that order and index of H are relatively prime. Show: the normalizer N of H in G is its own normalizer (i.e., $gNg^{-1} = N \Rightarrow gHg^{-1} = H$).

b). G = finite group. Show: A minimal subgroup of G is (simple or) direct product of simple groups that are pairwise isomorphic, in fact conjugate in G. (Hint: induction on the order.)

IV. Linear Algebra

1) a) T is an operator (= linear transformation) on a vector space V. In which of the following cases can one assert the existence of a (non-zero) eigenvector of T (proof or counterexample):

i) field = $\mathbb{C}$, dim $V = \infty$;

ii) field = $\mathbb{R}$, dim V finite, odd;

iii) field = $\mathbb{R}$, T skew-symmetric (with respect to some inner product in V).

b) A vector x_0 is cyclic for the operator T on V (dim $V = n < \infty$) if $\{x_0, Tx_0, T^2x_0, \dots\}$ span V. Show: If T has a cyclic vector, then the minimum polynomial of T is of degree n. What does the Jordan form of such a T look like?

2) a) Let A, B be two matrices over a field F. Suppose A and B are similar over an extension field K of F ($\exists$ P over K with $B = P^{-1}AP$). Prove: A and B are similar over F.

b) T is an arbitrary operator on a real vector space V (dim $V < \infty$). Show: There exists an invertible operator P such that TP is a projection (with what range?). Use this to show: Any right ideal in the algebra $\mathcal{L}$ of all operators on V is of the form $E\mathcal{L}$ with E a projection.