PH.D. QUALIFYING EXAMINATION

AUTUMN 1975-76

ALGEBRA

PART II

Attempt six (6) questions only. All questions carry equal weight. Answer each question in a separate blue book.

Notation:

$\mathbb{Z}$ = ring of integers

$\mathbb{Q}$ = field of rational numbers

$\mathbb{C}$ = field of complex numbers

1. Prove, from first principles, the following two isomorphism theorems:

   (i) If $H \supset K$ are two normal subgroups of a group G, then $(G/K)/(H/K) \simeq G/H$.

   (ii) If H, K are subgroups of a group G, with H contained in the normalizer of K, then $H/H \cap K \simeq HK/K$.

2. Find the Galois group of the splitting field of the polynomial $x^4 + 2$ over (i) $\mathbb{Q}$, (ii) the prime field with 3 elements, (iii) the prime field with 5 elements, (iv) the prime field with 73 elements. Briefly justify your answers.

3. Let A be an $n \times n$ matrix with coefficients in $\mathbb{C}$. Assume that each positive power of A has trace 0. Prove that $A^k = 0$ for some integer $k \geq 0$. (Hint: show that each eigenvalue must be 0 by induction on n.)

4. Let G be a cyclic group of prime order p, and K a field of characteristic p. Prove that one can define a representation of G in $GL_2(K)$ by mapping a generator of G to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Prove that this representation is not semi-simple.

5. Let L be the additive group $\mathbb{Z}^3$, and M the subgroup of L generated by the three vectors $x_1 = (1,2,3)$, $x_2 = (2,3,1)$, $x_3 = (3,1,2)$.

   (i) What is the index of M in L?

   (ii) Prove that M is not a direct summand of L.

   (iii) Is the subgroup generated by x_1 and x_2 a direct summand of L?

   (iv) What is the structure of L/M?

6. Let V be a finite dimensional vector space over a field, and let $B(x,y)$ be a non-degenerate bilinear form on V. Show that, if C is a bilinear form on V, there exists a unique map $L_C: V \to V$ such that $C(x,y) = B(L_C x, y)$ for all $x,y \in V$. Show that C is non-degenerate if and only if L_C is bijective. Show that there exists a unique isomorphism $P: V \to V$ such that $B(y,x) = B(Px,y)$ for all $x,y \in V$.

7. A complex number α is said to be constructible if there is a tower of fields

$$\mathbb{Q} = F_0 \subset F_1 \subset \cdots \subset F_n \subset \mathbb{C}$$

such that $\alpha \in F_n$ and F_{i+1}/F_i is a quadratic extension for all i. Prove that (i) $\cos \frac{2\pi}{17}$ is constructible, and (ii) $\sin \frac{\pi}{42}$ is not constructible.