Algebra Preliminary Examination September 21, 1992
1:00 - 4:00 p.m.

1. Let G be a finite group, K a normal subgroup and P a p-Sylow subgroup of K. Prove that $G = N_G(P) \cdot K$.

2. Let G be a p-group, that is, $|G| = p^m$ for some $m \in Z^+$. Prove that any maximal subgroup of G is normal and has index p. (Be sure to state what results about p-groups you are quoting.)

3. Let F be a free Abelian group with basis f_1, f_2, f_3, f_4 and let U be the subgroup of F generated by $u_1 = 2f_1 - f_2$, $u_2 = -f_1 + 2f_2 - f_3 - f_4$, $u_3 = -f_2 + 2f_3, u_4 = -f_2 + 2f_3$. Determine the structure of F/U. In particular, what is the rank of F/U and the isomorphism class of the torsion part of F/U?

4. Let R be a unique factorization domain (UFD) with multiplicative identity 1. Prove that if the greatest common divisor (gcd) of any $a, b \in R$ can be written as $ax+by$ for some $x, y \in R$, then R is a principal ideal domain (PID).

5. Let $L \supseteq K$ be an extension of fields and R a subring of L containing K, $L \supseteq R \supseteq K$.

   (a) Prove if L/K is algebraic then R is a field.

   (b) If L/K is not algebraic show that there exists an intermediate ring R which which is not a field.

6. Let V be a vector space over a field K and $f_1, \ldots, f_n$ linear functionals, i.e. K-linear maps $f_i : V \to K$. Prove that $f_1, \ldots, f_n$ are linearly independent in the dual space $V^* = Hom_K(V, K)$ if, and only if, $\bigcap_{i=1}^{n} \ker f_i$ has codimension n in V.

7. Let K be the splitting field over Q of $x^4 - 7$.

   (a) Prove $(K:Q) = 8$ and K is generated by i and $\sqrt[4]{7}$.
   (b) Prove that $Gal(K/Q) \cong D_8$, the dihedral group of order 8.

8. Let V be a vector space over a field K.

   (a) Define $\Lambda(V)$, the exterior algebra based on V.
   (b) Prove that every element of the algebra $\Lambda(V)$ is either nilpotent or invertible.