Algebra Preliminary Examination
September 20, 1993

1. Let K be a field and

$$J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

be a matrix with entries in K.
(a) Find and prove a formula giving the matrix J^n explicitly for all integers $n > 0$.
(b) Show that if p is a prime then $J^p = I$ if and only if K has characteristic p.

2. Consider a finite group with 6 Sylow 5-subgroups.
(a) Show that there is a group of order 60 with this property
(b) Show that no group of order less than 60 has this property.

3. Let G be the group of non-singular 2x2 matrices over the field with 2 elements, F_2. Let H be the symmetric group on 3 letters. Show that $G \cong H$.

4. Let $x_1, \ldots, x_n$ be n integers whose greatest common divisor is 1. Prove that there is an invertible nxn matrix M such that both M, M^{-1} have integer entries and such that the first column of M is $(x_1, \ldots, x_n)^T$.

Hint: Consider the free abelian group $\mathbb{Z}^n$ and let $x=(x_1, \ldots, x_n) \in \mathbb{Z}^n$.
(a) Show $\mathbb{Z}^n = \mathbb{Z}x \oplus A$ for some subgroup $A \subseteq \mathbb{Z}^n$
(b) Deduce the result from (a)

5. Let $g(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree p a prime, $p \geq 5$. Suppose that g(x) has exactly 2 nonreal roots in the field of complex numbers. Show that g(x) is not solvable by radicals.

6. Let $\alpha \in \mathbb{R}$ be algebraic over $\mathbb{Q}$, and let K be a splitting field over $\mathbb{Q}$ for the minimal polynomial of α over $\mathbb{Q}$. Suppose that the Galois group Gal $(K/\mathbb{Q})$ has cardinality a power of 2. Show that α is constructible.

7. Determine the Jordan canonical form for the nxn matrix over F_p whose entries are all equal to 1 (the answer depends on whether or not p divides n).

8. Let K be a field and let V be a vector space of dimension $n \geq 2$ over K. Let X be a one-dimensional subspace, H be a hyperplane (i.e. an $n - 1$ dimensional subspace) with $X \subseteq H$. A linear transformation $\sigma: V \rightarrow V$ is a <u>transvection</u> with center X and axis H if $\sigma | H = 1_H$, and $\sigma(v) - v \in X$ for all $v \in V$.

Let $f: V \times V \rightarrow K$ be a non-degenerate, alternating bilinear form. For a and $x \in V$ define $T_{a,x}: V \rightarrow V$ by $T_{a,x}(v) = v + af(x,v)v$.

(a) Prove that $T_{a,x}$ is a transvection with center $\langle x \rangle$ and axis$=\{v \in V | f(x,v) = 0\}$.

(b) Prove that $T_{a,x}$ is an isometry of f, that is for any $v, w \in V$, $f(v,w) = f(T_{a,x}(v), T_{a,x}(w))$.

(c) Let $a, b \in K^*$, $x, y \in V-\{0\}$. Prove that $T_{a,x}$ and $T_{b,y}$ commute if and only if $f(x,y) = 0$.