

4. Let p be a prime and F_p the field of p elements. For each integer $n \geq 1$, prove that there is at least one irreducible polynomial $p(x) \in F_p[X]$ of degree exactly n .
5. Compute the Galois group of the polynomial $x^8 - 1$ over \mathbb{Q} , explaining carefully the results that you use.
6. Let V be a finite-dimensional vector space over the complex numbers \mathbb{C} , equipped with a non-degenerate bilinear form $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{C}$ which is unitary (i.e. $\langle u, v \rangle = \overline{\langle v, u \rangle}$ for $u, v \in V$). Prove that V has a basis which is orthonormal with respect to $\langle \cdot, \cdot \rangle$.
7. Consider the matrix $A = \begin{pmatrix} 4 & 3 & 0 \\ -7 & -5 & 0 \\ 7 & 6 & 1 \end{pmatrix}$. Decide if there is matrix P such that $P^{-1}AP$ is upper triangular and with P having many entries in either (a) the real numbers, or, (b) the complex numbers.
8. Prove that there is no ring R (with an identity) whose underlying additive group is isomorphic to \mathbb{Q}/\mathbb{Z} .