Preliminary Exam, Algebra

June 11, 1999

1. Let G be a finite group, p a prime, and let P be a Sylow p-subgroup of G. Show that $N_G(N_G(P)) = N_G(P)$. (Here, $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$ denotes the normalizer of a subgroup H of G.)

2. Show that the rings $\mathbb{Z}/10\mathbb{Z}$ and $\mathbb{Z}[i]/(1+3i)$ are isomorphic.

3. Find a Jordan normal form of the $\mathbb{C}$-linear map

$$f\colon \mathbb{C}^4 \to \mathbb{C}^4, \quad \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \mapsto \begin{pmatrix} -\gamma - \delta \\ 8\alpha + 7\beta - 4\gamma - 10\delta \\ 10\alpha + 9\beta - 7\gamma - 17\delta \\ \delta \end{pmatrix}.$$

4. Let E/F be a Galois extension of degree 200.

(a) Show that there exists exactly one intermediate field $F \subset K \subset E$ such that $[K : F] = 8$.

(b) Show that K/F is again a Galois extension.

(c) Are there also intermediate fields of degree 2 and 4 over F? Can they be chosen to be Galois extensions over F?

5. Let V be a real vector space and let $\mathcal{J}\colon V \to V$ be an operator with $\mathcal{J}^2 = -\mathrm{id}_V$.

(a) Show that the definition $(a + bi)v := av + b\mathcal{J}v$, $a, b \in \mathbb{R}$, gives V the structure of a complex vector space.

(b) Show that an $\mathbb{R}$-linear operator $A\colon V \to V$ is also a $\mathbb{C}$-linear operator (with the complex structure from (a)), if and only if $A\mathcal{J} = \mathcal{J}A$.

6. Let V be a vector space over a field of characteristic not equal to 2 and let $f\colon V \to V$ be a linear map. Consider the linear map

$$f \wedge f\colon \Lambda^2 V \to \Lambda^2 V, \quad v \wedge w \mapsto f(v) \wedge f(w),$$

and show that the traces of f, $f^2 = f \circ f$, and $f \wedge f$ are related by $2\mathrm{tr}(f \wedge f) = \mathrm{tr}(f)^2 - \mathrm{tr}(f^2)$.