GRADUATE PRELIMINARY ANALYSIS EXAM

September 22, 1980

1. Let $f(z) = \sum_{i=0}^{\infty} a_i z^i$ be an entire function. Assume that there exists a constant $M > 0$ and an integer $m \geq 0$ such that $|f(re^{i\theta})| \leq M\gamma^m$ for all sufficiently large γ. Show that $f(z)$ is a polynomial of degree no higher than m.

2. Let a be a complex number with $a > 0$ and let $\mathfrak{F}$ be the set of all functions analytic in $|z| < 1$ and satisfying the conditions $\operatorname{Re} f(z) > 0$ and $f(0) = a$. Show that the set $\{|f'(0)| : f \in \mathfrak{F}\}$ has a maximum. Determine its value and the members of $\mathfrak{F}$ for which the maximum is achieved.

3a. State carefully the Cauchy Integral Theorem.

b. Compute $\int_{-\infty}^{\infty} \frac{x}{\sinh x} dx$ $\quad (\sinh z = \frac{e^z - e^{-z}}{2})$

4a. Define each of the following terms for a topological space

i) compact
ii) connected
iii) Hausdorff
iv) perfect

b. For each property P from (i)-(iv), either prove that the other three properties imply P or give an example which shows that the other three properties do not imply P.

5. Let $\{x_n\}$ be a sequence of points in $[0,1]$ such that $x_n \to 0$. There exists a nonnegative C^∞ function on $[0,1]$ with $f(x_n) = 0$ for all n and $f(x) > 0$ if x is not one of the x_n. True or false? Prove your assertion.