Fall 1998

Analysis Preliminary Exam

September 30, 1998
12:00 - 5:00pm

1. Show that if A is a bounded measurable set then $\lim_{n\to\infty}\int_A \sin nx\,dx = \lim_{n\to\infty}\int_A \cos nx\,dx = 0$.

2. Show that if $\sum_{n=1}^{\infty}\int |f_n(x)|\,dm(x) < \infty$ then the series $\sum_{n=1}^{\infty} f_n(x)$ converges for almost every x.

3. Consider the set $\mathbf{R}^{\omega}$ of all sequences $\{a_n\}_{n=1}^{\infty}$ of real numbers and let C denote the subset of all sequences such that $\sum_{n=1}^{\infty} a_n$ converges. Is the function $\{a_n\} \to \sum a_n$ continuous from C to $\mathbf{R}$ when C is given (a) the product topology? (b) the box topology? Justify your answers.

4. Prove that a compact metric space is complete.

5. Let f be a holomorphic function in the disc $|z| < 1$. For $r \in [0,1)$, set $M(r) = \max_{|z|=r}|f(z)|$.

   (a) Show that M is increasing (in the broad sense).

   (b) Assume that $M(r) \le r$ for some $r \in [0,1)$. Prove that $M(\rho) \le \rho$ for all $\rho \in [0,r]$.

6. Let $F(\theta_1,\theta_2)$ be a C^1 function on $S^1 \times S^1$ (product of two circles).

   (a) Show that the operator T_F given by $(T_F\varphi)(\theta_1) = \int_{S^1} F(\theta_1,\theta_2)\varphi(\theta_2)\,d\theta_2$ is a trace class operator on $L^2(S^1 \times S^1)$

   (b) Compute the trace of T_F.

   Hint: Use the Fourier series expansion of $F(\theta_1,\theta_2)$ with respect to $\{e^{2\pi i n\theta_1}e^{2\pi i m\theta_2} \mid n,m \in \mathbf{Z}\}$.

7. Give an example of Hilbert Schmidt operator which is not of trace class.

8. Let n be an interger ≥ 2. Show that $\int_o^\infty \frac{dx}{1+x^n} = \frac{\pi/n}{\sin(\pi/n)}$, by integrating the function $\frac{1}{1+z^n}$ along the contour formed by the segment [0,R] of the positive real axis, the arc represented by $\mathrm{Re}^{it}, 0 \le t \le \frac{2\pi}{n}$ and the segment represented by $re^{\frac{2\pi i}{n}}, 0 \le r \le R$.