

1. Show that if  $A$  is a bounded measurable set then  $\lim_{n \rightarrow \infty} \int_A \sin nx dx = \lim_{n \rightarrow \infty} \int_A \cos nx dx = 0$ .
2. Show that if  $\sum_{n=1}^{\infty} \int |f_n(x)| dm(x) < \infty$  then the series  $\sum_{n=1}^{\infty} f_n(x)$  converges for almost every  $x$ .
3. Consider the set  $\mathbf{R}^{\omega}$  of all sequences  $\{a_n\}_{n=1}^{\infty}$  of real numbers and let  $C$  denote the subset of all sequences such that  $\sum_{n=1}^{\infty} a_n$  converges. Is the function  $\{a_n\} \rightarrow \sum a_n$  continuous from  $C$  to  $\mathbf{R}$  when  $C$  is given (a) the product topology? (b) the box topology? Justify your answers.
4. Prove that a compact metric space is complete.
5. Let  $f$  be a holomorphic function in the disc  $|z| < 1$ . For  $r \in [0, 1)$ , set  $M(r) = \max_{|z|=r} |f(z)|$ .
  - (a) Show that  $M$  is increasing (in the broad sense).
  - (b) Assume that  $f(0) = 0$  and  $M(r) \leq r$  for some  $r \in [0, 1)$ . Prove that  $M(\rho) \leq \rho$  for all  $\rho \in [0, r]$ .
6. Let  $F(\theta_1, \theta_2)$  be a  $C^1$  function on  $S^1 \times S^1$  (product of two circles).
  - (a) Show that the operator  $T_F$  given by  $(T_F \varphi)(\theta_1) = \int_{S^1} F(\theta_1, \theta_2) \varphi(\theta_2) d\theta_2$  is a trace class operator on  $L^2(S^1 \times S^1)$ .
  - (b) Compute the trace of  $T_F$ .

Hint: Use the Fourier series expansion of  $F(\theta_1, \theta_2)$  with respect to  $\{e^{2\pi i n \theta_1} e^{2\pi i m \theta_2} | n, m \in \mathbf{Z}\}$
7. Give an example of Hilbert Schmidt operator which is not of trace class.
8. Let  $n$  be an integer  $\geq 2$ . Show that  $\int_0^{\infty} \frac{dx}{1+x^n} = \frac{\pi/n}{\sin(\pi/n)}$ , by integrating the function  $\frac{1}{1+z^n}$  along the contour formed by the segment  $[0, R]$  of the positive real axis, the arc represented by  $R e^{it}, 0 \leq t \leq \frac{2\pi}{n}$  and the segment represented by  $r e^{\frac{2\pi i}{n}}, 0 \leq r \leq R$ .