Fall 1999

Preliminary Exam, Algebra

September 22, 1999

1. Let p be a prime, $n \in \mathbb{N}$, and G a group of order p^n. Show that every subgroup of index p in G is normal in G.

2. An element e of a ring R is called an *idempotent*, if $e^2 = e$.
(a) Let p be a prime and $n \in \mathbb{N}$. Show that 0 and 1 are the only idempotents in $\mathbb{Z}/p^n\mathbb{Z}$.
(b) Determine all idempotents of $\mathbb{Z}/1000\mathbb{Z}$.

3. Consider the subset

$$U := \{(w, x, y, z) \in \mathbb{Z}^4 \mid x - 2y + 6z = 0 \text{ and } z \in 5\mathbb{Z}\} \subseteq \mathbb{Z}^4.$$

(a) Show that U is a subgroup of $\mathbb{Z}^4$.
(b) Find a $\mathbb{Z}$-basis of U.
(c) Determine the isomorphism type of $\mathbb{Z}^4/U$.

4. Find an element $\alpha \in \mathbb{C}$ such that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is a Galois extension of degree 5.

5. Let A be an orthogonal operator acting on the 5-dimensional Euclidean space $E^5 = \mathbb{R}^5$. Suppose that the characteristic polynomial of A has two distinct roots $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$ such that $\bar{\lambda} \neq \mu$. Show that there exist two orthogonal 2-dimensional subspaces E_1 and E_2 of E such that $A(E_i) = E_i$, for $i = 1, 2$.

6. Let V be a vector space of dimension $n \geq 2$ and let $u \in \Lambda^2 V$. Show that u can be written as $e \wedge f$ for some $e, f \in V$ if and only if $u \wedge u = 0$ in $\Lambda^4 V$. (Hint: Use induction on n and use a basis $\{e_1, \ldots, e_n\}$ of V. For the induction step $n \to n+1$, write $u = e_{n+1} \wedge v + w$ with suitable $v \in V$ and $w \in V \wedge V$. Show that $v \wedge w = 0$ and $w \wedge w = 0$ and apply the induction hypothesis to w.)