Fall 1999

Mathematics Analysis Preliminary Exam
December 10, 1999

1. Does every metric space have a complete metric eqivilant to the original one (i.e. given the same topology)? Give a proof if true and a counterexample otherwise.
(Hint: use Baire's theorem).

2. a)Give an example of an uncountable set of Lebesgue measure zero.

b) Construct a closed set of uncountable irrational numbers.

3. Suppose that f_n is a sequence of measurable functions on a complete measure space (X, Λ, μ) such that $f_n \to f$ in measure μ and suppose that $| f_n(x) | \leq g(x) \qquad \mu - a.e.$ for some $g \in L^1_\mu(X)$. Then

$$\lim_{n \to \infty} \int_X f_n d\mu = \int_X f d\mu$$

4. Let f be a measurable function on X such that $\|f\|_r < \infty$ for some $r < \infty$. Prove that

$$\|f\|_p \to \|f\|_\infty \qquad as \qquad p \to \infty$$

Hint:
First show that $lim\|f\|_p \leq \|f\|_\infty$ by observing that $\|f\|_p \leq \|f\|_r^\alpha \|f\|_\infty^\beta$ for some α and β. Consider the set $B = \{x \mid |f(x)| + \epsilon \geq \|f\|_\infty\}$ to prove the opposite inequality.

5. Let f be meromorphic on a neighborhood of zero. Prove that

$$\max_{|z|=R} |f(z)| \geq \frac{|a_{-1}|}{R},$$

where $\sum a_n z^n$ is the Laurent expansion of f.

6. Suppose f is analytic except for a finite number of isolated singularities and that

$$\lim_{z \to \infty} z f(z) = 0.$$

Prove that the sum of residues of f is equal to zero.

7. Let A be an operator in $l_2(\mathbb{R})$ given by the formula $A(\{x_n\}) = \{a_n x_n\}$ where $\{a_n\}$ is a bounded sequence.

(a) Prove that $a_n \to 0$ if A is compact.

(b) Prove the converse of (a): if $a_n \to 0$ then A is compact.