ALGEBRA

Write your name, the part number, and the question number on each blue book.

Part I. Miscellaneous questions [give brief reasoning]. (20%) Write answers to 1, 2, 3 in one blue book, and to 4, 5 in a separate blue book.

1. Construct an integral domain with 6 elements, or prove that no such thing exists.

2. Let $\varphi: R \to S$ be a homomorphism of rings (with unit). Prove or give counterexamples for the following two statements: (1) If J is a maximal ideal of S, then $\varphi^{-1}(J)$ is a maximal ideal of R; (2) same as (1) with "prime" replacing "maximal".

3. Suppose $g \to g^{-1}$ is an automorphism of a group G. Is G necessarily Abelian?

4. Let $f(x)$ be a polynomial over a field K. Suppose $f(a)$ is 0 for all elements a of K. Does it follow that f is the polynomial 0?

5. Let A be an algebra (not necessarily associative) of finite dimension over a field K, without zero-divisors. Prove: for $a \neq 0$ and any b in A the equation $a \cdot x = b$ has a solution in A.

Part II. Rings, fields. Do one of 1, 2, 3 (10%), and one of 4, 5, 6 (20%). Use a separate blue book for each answer.

1. Let A be a ring (with 1), M an A-module (with $1 \cdot m = m$), let B be the ring of A-endomorphisms of M. Prove Schur's lemma:

   a) If M is simple (irreducible), then B is a division ring.

   b) If furthermore M is a finite dimensional vector space over $\mathbb{C}$, and A consists of $\mathbb{C}$-linear maps, then $B = \mathbb{C}$.

2. Let R be a commutative ring with 1; let $x_1, \ldots, x_n$ be indeterminates. Let G be the group of automorphisms of the polynomial ring $R[x_1, \ldots, x_n]$ generated by the permutations of the x_i and all the changes of sign $x_i \to -x_i$. Describe the elements invariant under G.

3. K is a field of char. p; x, y two indeterminates. Show that $K(x^{1/p}, y^{1/p})$ does not have a primitive element over $K(x,y)$ (i.e., cannot be generated by a single element).

4. k is a field, x an indeterminate. Construct all (non-archimedean) valuations on the field $k(x)$ that are trivial on k.

5. Let A, M, B be as in #1. Let C be the ring of B-endomorphisms of M (double commutator of A; note: the maps $m \to a \cdot m$, for fixed a in A, belong to C).

   a) Show: if N is direct summand of M (as A-module), then N is C-invariant. [Hint: consider a projection of M onto N].

   b) Suppose M is semi-simple [every A-submodule has a complement]. Use a) to show: To m in M and f in C there exists a in A with $f(m) = a \cdot m$.

   c) More generally, show: If $\{m_1, \ldots, m_n\}$ is a finite subset of M, and f in C, then there exists a in A with $f(m_i) = a \cdot m_i$, $i = 1, \ldots, n$. [Hint: consider $M \oplus \cdots \oplus M$, with the diagonal action of A, and describe the double commutator.]

6. Define the notion of primitive n-th root of unity (over the rationals, say). Let r_n be the number of primitive $\sqrt[n]{1}$'s Show that $r_n = \varphi(n)$ (Euler φ-function) and that for each n the multiplicative group of Show that <u>all</u> $\sqrt[n]{1}$'s is cyclic. Define the n-th cyclotomic polynomial $\Phi_n(x)$; $\prod_{d|n} \Phi_d(x) = x^n - 1$, and that $\Phi_n(x)$ belongs to $Z[x]$.

Part III. Groups. Do one of 1, 2, 3 (10%) and one of 4, 5 (20%). Use a separate blue book for each answer.

1. S_n = symmetric group on n objects; G a subgroup of S_n of order p^k, where p is a prime not dividing n. Show G has a fixed point (one of the n objects is left fixed by every element of G).

2. Let G be a finite group, N a normal subgroup, H the quotient group G/N. Show: G is solvable if and only if N and H are so.

3. G a group; K, L two subgroups.

   a) Show: the intersection of a left coset of K and a left coset of L is either empty or a left coset of $K \cap L$. Give an example where "empty" occurs.

   b) Show: if K and L have finite index in G, so does $K \cap L$.

4. An Abelian group A is <u>divisible</u> if $a = n \cdot x$ has a solution for every a in A and every positive integer. A is <u>injective</u> if given any Abelian group E, a subgroup F of E, and a homomorphism $\varphi: F \to A$, one can extend φ to $\bar{\varphi}: E \to A$. Show: A is divisible iff it is injective.

5. Let G be a finite group; we consider representations of G on finite dimensional vector spaces over $\mathbb{C}$. Define the notion of matrix element of a representation; state the orthogonality relations; explain how one can read off the number of inequivalent irreducible representations from the group theoretical structure of G.

Part IV Linear Algebra. Do one of 1, 2, 3 (10%) and one of 4, 5 (10%). Use a separate blue book for each answer.

V is a vector space of finite dimension over a field F; $A, B, \ldots$ are operators on V (or matrices if you wish).

1. $F = \mathbb{R}$. Suppose V has an inner product defined. Show: If A is positive definite, and B commutes with A^2, then B commutes with A.

2. $F = \mathbb{C}$. Show: If each A_α of the family $\{A_\alpha\}$ can be diagonalized and any two A_α commute, then there is a simultaneous diagonalization.

3. $F = \mathbb{R}$. Define an inner product on the space of matrices by $(A,B) = tr(A \cdot B')$ (here tr = trace, and B' is the transpose of B). Find the orthogonal complement of the subspace of skew symmetric matrices.

4. For a matrix A over F, how can one decide whether A can be diagonalized (in a suitable overfield of F). Which overfield of F, if any, does one need to consider?

5. $F = \mathbb{C}$.

   a) Show: If A is nonsingular, then there exists a polynomial $p(x)$ without constant term, such that $p(A) = I$.

   b) Let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of A; let $V_1, \ldots, V_k$ be the generalized eigenspaces (v belongs to V_i if it is annihilated by some power of $A - \lambda_i$). Then $V = V_1 \oplus \cdots \oplus V_k$; let P_i be the projections associated with this decomposition.

Prove: there exist polynomials $q_i(x)$ with $q_i(A) = P_i$. [Note: by a) is it enough to find $r_i(x)$ with $r_i(A)$ nonsingular on V_i and 0 on the other V_j.]