ALGEBRA

Parts I and II

<u>Answer each question in a separate blue book</u>. Write your name, the Part number, and the Question number on the cover of each blue book.

Part I: <u>Linear Algebra</u> (25%)

<u>Do two problems</u>.

1. Let dim V be finite, V^* the dual of V, $L(V)$ the set of all linear transformation from V into V. Define a "natural" map from $V \otimes V^*$ onto $L(V)$ and show it is an isomorphism.

2. (From first principles) If a system of linear equations

$$\sum_{j=1}^{n} a_{ij}x_j = b_i ,$$

with a_{ij}, b_i in a field F, can be solved in some overfield $K \supset F$, it can be solved in F.

3. Let char $F \neq 2$, $B(x,y)$ a non-degenerate skew-symmetric bilinear form over a vector space V, scalars F, dim V finite. Show dim V is even and there is a basis $e_1, \ldots, e_n$, $f_1, \ldots, f_n$ for V such that

$$B(e_i,e_j) = 0 , B(f_i,f_j) = 0 , B(e_i,f_i) = 1 , \text{ and if } i \neq j , B(e_i,f_j) = 0 .$$

Part II: Groups (25%)

Do two problems.

1. (From first principles) Let G be a torsion abelian group. $G_p = \{x \mid p^k x = 0 \text{ for some } k\}$. Show G is the direct sum of all the G_p.

2. Let G be the group of all permutations of the integers. Give an example of a proper normal subgroup of G and justify.

3. (From first principles) Let $n = p^a m$, $p \nmid m$. If G is of order n, H a normal subgroup of order p^a, then H contains all subgroups K where the order of K is a power of p.

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Parts III and IV

Part III. Fields (25%)

Do problems 1 and either 2 or 3.

1. Let L/k be a normal extension of finite degree and K/k an arbitrary extension. Assume L and K are contained in a composite field KL and $K \cap L = k$. Show that KL/K is normal and that the Galois group of KL/K is isomorphic to that of L/k.

2. Let k be a subfield of the real numbers. Let K be an extension field of k, with $[K:k]$ odd. Show that the equation $x_1^2 + \dots + x_n^2 = -1$ cannot be solved in K.

3. Give an example of fields $k \subset K \subset L$, with K/k normal, L/K normal but L/k not normal, and justify.

Part IV. Rings (25%)

Do two problems.

1. State the decomposition theorem for ideals in a commutative Noetherian ring, including uniqueness statements.

Give an example of an integral domain which is not Noetherian and prove.

2. Let k be a field, A_n the ring of $n \times n$ matrices over k. Express A_n as a direct sum of minimal left ideals and prove.

3. Let R be an integral domain (commutative) in which every ideal is generated by one element. Show that (i) R satisfies the ascending chain condition for ideals, and (ii) every element can be uniquely factored up to order and units.