Ph.D. QUALIFYING EXAMINATION AUTUMN 1969-70

ALGEBRA

Parts I and II

All questions have equal weight.

Answer each question in a separate blue book. Write your name, the Part number, and the Question number on the cover of each blue book.

Notation : $\mathbb{Z}$ = integers; $\mathbb{Z}_n$ = cyclic group of order n; $\mathbb{Q}$ = rationals; $\mathbb{R}$ = reals; $\mathbb{C}$ = complex numbers; $\mathbb{C}^* = \mathbb{C} - \{0\}$.

Part I: Groups (25%)

Do 1 or 2 and 3 or 4.

1. Let Γ be the (multiplicative) subgroup of $\mathbb{C}^*$, consisting of all n-th roots of 1, for all n. Show:

   a) Γ has for each n exactly one subgroup of order n; describe the structure of the subgroup.

   b) If Φ is any finite subgroup of Γ, then the factor group Γ/Φ is isomorphic to Γ.

   c) For each prime p, the p-primary part Γ_p of Γ has no proper ($\neq \Gamma_p$) infinite subgroup.

2. a) Define "rank of an Abelian group" and state a relation between the ranks of a group, a subgroup and the corresponding factor group.

   b) Show that a finitely generated Abelian group A cannot be isomorphic to any factor group A/B with non-zero B

3. a) Prove that a finite p-group has non-trivial center.

   b) State some Sylow theorems.

4. Classify the irreducible representations (finite dimensional, over $\mathbb{C}$) of a finite cyclic group $\mathbb{Z}_n$. Sketch proof.

Part II: Rings (25%)

Do 1 or 2 and 3 or 4.

1. R = commutative ring with 1; $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ maximal ideals, pairwise different. Show:

   The natural map $R/\bigcap \mathfrak{p}_i \longrightarrow \Pi_1^n R/\mathfrak{p}_i$ is an isomorphism.

   (Find elements $(0,\ldots,a,\ldots,0)$ in the image.)

2. Let Λ be a lattice in $\mathbb{C}$, i.e., a finitely generated (and thus free) additive subgroup ($\neq 0$). Show: A complex number z that satisfies $z \cdot \Lambda \subset \Lambda$ is an algebraic integer. ($z \cdot \Lambda = \{z \cdot \lambda;\ \lambda \in \Lambda\}$.)

3. a) Define "primary ideal" and "associated prime ideal" for commutative rings; prove that the latter are indeed prime.

   b) Find a primary decomposition and the associated primes for the ideal $(9, 3x)$ in $\mathbb{Z}[x]$.

4. Show: If the ring R is sum of a family (J_α) of minimal left ideals, then every left R-module M with $R \cdot M = M$ is semisimple, i.e., direct sum of irreducible submodules.

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Parts III and IV

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Part III: Fields (25%)

Do 1 or 2 and 3 or 4.

1. a) Define "transcendence degree of an extension $K \subset L$" and state the relevant law for $K \subset L \subset M$.

   b) Let p_1, p_2, p_3 be three polynomials in 2 variables u, v with real coefficients. Show: The surface in $\mathbb{R}^3$, given by $x_i = p_i(u,v)$, $u, v \in \mathbb{R}$, is algebraic in the sense that all these points (x_1, x_2, x_3) satisfy some algebraic equation $q(x_1, x_2, x_3) = 0$ (where q has real coefficients).

2. a) Show: If the field K has char $= p > 0$, then for any a in K either a has a p-th root in K or $x^p - a$ is irreducible over K.

   b) Proof or counter example for: An algebraic extension of a perfect field is perfect.

   c) (see next page)

2. c) Same for: A purely transcendental extension of a perfect field is perfect.
(A field is perfect if every irreducible polynomial over it is separable.)

3. Let K be the splitting field of $x^5 - 3 = 0$ over $\mathbb{Q}$, with the Galois group G over $\mathbb{Q}$. Let P be the field of 5-th roots of 1 over $\mathbb{Q}$; denote by F, respectively H, the Galois group of P over $\mathbb{Q}$, respectively K over P.

   a) Show that F and H are cyclic, and find their orders. Describe generators s,t of F and H in terms of their action on suitable elements.

   b) Show (by group theory) that the extension $1 \to H \to G \to F \to 1$ splits, i.e., that G has a subgroup E that projects isomorphically onto F.

   c) Determine the extension, i.e., letting r,t denote suitable generators of E and H, find $r^{-1} \cdot t \cdot r$.

4. Let $\alpha_1, \ldots, \alpha_n$ be the roots (in $\mathbb{C}$, say) of an irreducible polynomial over $\mathbb{Q}$. Let $g(x)$ be any polynomial in $\mathbb{Q}[x]$, and put $\theta_i = g(\alpha_i)$, $i = 1, \ldots, n$. (The θ_i are not necessarily distinct.) Show:

   a) (by Galois considerations) the <u>different</u> elements that occur in the sequence $(\theta_1, \ldots, \theta_n)$ appear all equally often.

   b) The polynomial $h(x) = \Pi_1^n (x - \theta_i)$ lies in $\mathbb{Q}[x]$.

   c) If $r(x)$ is the monic irreducible polynomial for θ_1 over $\mathbb{Q}$, then $r(\theta_i) = 0$ holds for all $i = 1, \ldots, n$.

   d) $h(x) = r(x)^{n/m}$, where $m = \deg r(x)$.

Part IV: Linear Algebra (25%)

Do 1 or 2 and 3 or 4.

1. Let V be a vector space over $\mathbb{R}$ ($\dim V \leq \infty$); let $b(x,y)$ be a bilinear form on V. Let W be a finite dimensional subspace such that the restriction of b to W is non-degenerate. Show: $V = W \oplus W^{\perp}$. (Here $W^{\perp} = \{x \in V: b(x,y) = 0, \forall\ y \in W\}$.)

2. V = finite dimensional vector space over $\mathbb{C}$; T an operator on V; let $\mathcal{J} = \mathbb{C}[T]$ be the algebra formed by all polynomials in T. Show: T is semisimple (= diagonizable) iff $\mathcal{J}$ is semisimple (= has no nilpotent elements). (Use any standard facts for operators.)

3. $A = [a_{ij}]$ an $n \times n$ matrix over $\mathbb{R}$. Suppose: $a_{ii} > 0$, $a_{ij} < 0$ for $i \neq j$, $\sum_j a_{ij} > 0$ for all i. Show that A is non-singular. (Show directly that there is no dependence relation between the columns.)

4. V = finite dimensional vector space. Show:
   a) If A, B are two operators whose kernels N_A and N_B are identical, then there exists an invertible operator P with $B = P \cdot A$. (Consider a complement to N_A.)
   b) If A, B are two operators with $N_A \subset N_B$, then there exists an operator Q with $B = Q \cdot A$.

   (Note: by a) one may replace A and B by any operators with the given null spaces.)