Ph.D. QUALIFYING EXAMINATION AUTUMN 1970-71

ALGEBRA

Parts I and II

<u>All questions have equal weight</u>.

<u>Answer each question in a separate blue book</u>. Write your name, the Part number, and the Question number on the cover of each blue book.

<u>Notation</u>: $\mathbb{Z}$ = integers; $\mathbb{Z}_n$ = cyclic group of order n ;

$\mathbb{R}$ = reals; $\mathbb{C}$ = complex numbers; $\mathbb{Q}$ = rationals.

Part I: Groups (25%)

1. Let G be a finite group, and let H be a subgroup of G . Show that the union of the conjugates of H by elements of G cannot be all of G .

2. Let G be a finite p-group (p a prime). Let H be a normal subgroup of G . Show that H has non-trivial intersection with the center of G .

3. Provide proof or counterexamples to the following statements:

   a) If G is a finite group with normal p-Sylow subgroup P , then P is a direct summand of G .

   b) If G is a group with normal subgroups K and L such that $G/K \cong G/L$, then $K \cong L$.

Part II: Linear Algebra (25%)

1. Let V be an n-dimensional vector space over $\mathbb{R}$. Let T be a linear transformation on V, and suppose that $T^2 = -I$ (I is the identity transformation). Show that n is even.

2. Let G be the group of $n \times n$ invertible matrices over the field K. Let H be the subgroup of G consisting of matrices of determinant ± 1.

   a) Show that H is a normal subgroup of G.

   b) If $K = Z_p$, p a prime, find the order of G/H.

3. Let V be a finite dimensional vector space over the field K. Let E be the vector space over K of linear transformation from V to V. Let $A \in E$. Define

$$R_A : E \to E \qquad \text{by}$$

$$R_A(B) = BA, \text{ for each } B \in E.$$

Describe the characteristic roots of R_A in terms of those of A.

Ph.D. QUALIFYING EXAMINATION AUTUMN 1970-71

ALGEBRA

Parts III and IV

All questions have equal weight.

Answer each question in a separate blue book. Write your name, the Part number, and the Question number on the cover of each blue book.

Notation: $\mathbb{Z}$ = integers; $\mathbb{Z}_n$ = cyclic group of order n ;

$\mathbb{R}$ = reals; $\mathbb{C}$ = complex numbers; $\mathbb{Q}$ = rationals.

Part III: Fields (25%)

1. Let k be a field, and let $f(x)$ be an irreducible separable polynomial in $k[x]$. Let K be the splitting field of $f(x)$ over k , and assume that the Galois group of K over k is abelian. Prove that any root of $f(x)$ generates K over k .
2. Consider the polynomial $f(x) = x^8 + x^4 + 1$ over $\mathbb{Z}_2$. Let K be its splitting field. Find:

   a) the degree of K over $\mathbb{Z}_2$;

   b) the order of the Galois group of K over $\mathbb{Z}_2$.
3. Let $f(x)$ be a monic polynomial in $\mathbb{Z}[x]$ with roots $r_1 , \dots , r_n$ in $\mathbb{C}$. Suppose that $|r_i| = 1$, for each $i = 1 , \dots , n$ (where $|\,|$ denotes absolute value). Let $g_m(x) = \sum_{i=0}^{n} b_{im}x^i$ be the

polynomial with roots $r_1^m, \ldots, r_n^m$, for any non-negative integer m.

a) Show that $|b_{im}| \leq \binom{n}{i}$, for each i, independent of m.

b) Show that $g_m(x) \in \mathbb{Z}[x]$, for each m.

c) Using the polynomials $g_m(x)$, or otherwise, prove the following: If a monic polynomial in $\mathbb{Z}[x]$ has all its roots on the unit circle in the complex plane, then all its roots are roots of unity.

Part IV: Rings (25%)

1. Let R be a commutative ring with 1 such that every additive subgroup of R is an ideal of R. Prove that R is isomorphic to $\mathbb{Z}$ or to $\mathbb{Z}_n$ for some natural number n.

2. Let R be a commutative ring with 1 satisfying the descending chain condition on ideals. Assume that the intersection of all the maximal ideals is $\{0\}$. Show R is a direct sum of finitely many fields.

3. Let R be a commutative ring with 1, and let J be the intersection of all the maximal ideals of R. Prove that:

   a) $1+m$ is a unit of R, for any $m \in J$;

   b) if I is an ideal such that $m \in I$ implies $1+m$ is a unit, then I is contained in J.