(2) (35 pts) Let V_1 have basis e_1, e_2, e_3 and let $A: V_1 \to V_1$ be given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Let V_2 have basis f_1, f_2 and $B: V_2 \to V_2$ be given by

$$\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}.$$

Find the Jordan canonical form for $A \otimes B: V_1 \otimes V_2 \to V_1 \otimes V_2$, and indicate your steps.

(Hint: find the ranks of $(A \otimes B - 2I)^i$.)

(3) (30 pts) Let A be a 2×2 matrix with entries in $\mathbb{C}$. Prove the number of square roots of A is 0, 2, 4 or ∞.

(4) (35 pts) Let V be a finite dimensional vector-space over the the field Z_2, and let

$$A: V \to V^*$$

represent a symmetric, non-degenerate, bilinear form on V. Thus A is an isomorphism and

$$[A(y)](x) = [A(x)](y)$$

for any two vectors x, y in V.

(i) Show there is a unique element $\upsilon \in V$ so

$$[A(\upsilon)](x) = [A(x)](x)$$

for all $x \in V$.

(ii) Prove $[A(\upsilon)]\upsilon = 1$ if and only if the dimension of V is odd.

(Hint: Show there is a basis for V so A has the form $\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}$

where B has zeroes on the diagonal.)

Part II: Groups

Do any three problems.

(1) (20 pts) Let G be a finite group all of whose Sylow subgroups are normal. Prove G is the direct product of its Sylow subgroups.

(2) (20 pts) Let N be a finite normal subgroup of the group G, and let C be the centralizer of N in G. Show that C is normal and has finite index in G.

(3) (30 pts) Prove (without using Burnside's theorem) that all groups of order 24 and 36 are solvable.
(The Sylow theorems are not quite enough here.)

(4) (30 pts) Consider the automorphism group of the direct product $Z_5 \times Z_5$.

(i) Find its order.

(ii) Find a Sylow 2-subgroup and describe its generators and relations. Indicate proofs.

(5) (40 pts) Calculate the number and the dimensions of the irreducible complex representations of S_4, the symmetric group on 4-letters.
(Include proofs or quote the theorems used.)

Ph.D. Qualifying Examination

ALGEBRA

Autumn 1971-72

Parts III and IV

The points assigned indicate the approximate difficulty of the problems. Answer each question in a separate blue book. Write your name, the part number, and the question number on the cover of each blue book.

Notation
$\mathbb{Z}$ = integers
$\mathbb{Z}_n$ = cyclic group of order n .
$\mathbb{R}$ = Reals
$\mathbb{C}$ = complex numbers
$\mathbb{Q}$ = rationals
$A[x]$ = set of polynomials in x with coefficients in A.

It is to your advantage to work a few problems completely rather than attempting to obtain partial credit on many.

Part III: Rings

Do two from (1), (2), (3) and one of (4), (5).

(1) (25 pts) Let A be a commutative ring with unit. An A-module N is Noetherian if every A-submodule of N is finitely generated. Prove that if

$$0 \to M \to N \to P \to 0$$

is an exact sequence of A-modules, then N is Noetherian if and only if M and P are.

(2) Let A be a commutative ring with unit, and let $f(x) \in A[x]$ be a monic polynomial of degree n.

(i) (15 points) Assume there are n elements $a_1, \ldots, a_n$ in A so that $f(a_i) = 0$, $i = 1, \ldots, n$ and no difference $a_i - a_j$ is a zero divisor in A for $i \neq j$. Show

$$f(x) = (x-a_1)\cdots(x-a_n)$$

(ii) (10 points) Give an example to show the assumptions of (i) are necessary.

(3) (25 pts) Prove that in a commutative ring A with unit every maximal ideal is prime. Give an example (with proof) of a non-zero prime ideal in some ring A with unit which is not maximal.

(4) (20 points) (i) Let $A = Z[\sqrt{-1}]$. Show A is a principal ideal domain.

(15 points) (ii) Let $A = Z[\sqrt{-5}]$. Show A is not a unique factorization domain. Indicate proofs.

(5) Consider the complex group ring $\mathbb{C}(G)$ for a finite group G:
($\mathbb{C}(G)$ is the set of all formal sums $\Sigma\, c_r g_r$ for $c_r \in \mathbb{C}$, $g_r \in G$ with obvious sum rule and product

$$\Sigma\, c_r g_r \,\Sigma\, d_s g_s = \Sigma\, c_r d_s g_r g_s\ .)$$

It is a theorem that $\mathbb{C}(G)$ is semi-simple.

(20 points) (i) Identify the types and numbers of each type of simple algebra M in a decomposition of $\mathbb{C}(G)$ into a direct sum of simple algebras where G is the dihedral group on generators x, y with relations $x^2 = y^3 = 1$, $xyx = y^2$.

(25 points) (ii) Corresponding to each simple algebra M found in part (i) construct an idempotent I_M in $\mathbb{C}(G)$ so $\mathbb{C}(G) \cdot I_M \cong M$. (Hint: If M is one dimensional and $g \in G$, what can you say about $gI_M = \theta I_M$.)

Part IV: Fields

Do number (1) and one from (2), (3) and (4).

(1) (20 pts) Let F be a real field (-1 is not a sum of squares in F). Prove that if A is a sum of squares in F then $F(\sqrt{A})$ is again a real field.

(2) (40 pts) What groups occur as the Galois groups of irreducible cubics over Q? Give an example of cubic with each such group. (Include proofs.)
(As an aid note that the discriminant D for the cubic $x^3 + px + q$ is $-4p^3 - 27q^2$.)

(3) (30 points) Prove that a regular pentagon can be constructed with only a straight edge and compass, using the Galois theory of fields, and indicate a method for finding the actual constructions necessary.

(4) (10 pts) (i) Let F be an algebraic extension field of Q. Define the ring A of algebraic integers in F.

(15 pts) (ii) Let n be a positive integer and α, β any two primitive n^{th} roots of unity. Show

$$\frac{1 - \alpha}{1 - \beta}$$

is an algebraic integer in $Q(\alpha)$.

(20 pts) (iii) Let p be a prime, ρ a primitive p^{th} root of unity, and A the ring of algebraic integers in $Q(\rho)$. Show there is a principal ideal $\mathfrak{J} \subset A$ so $\mathfrak{J}^{p-1} = (p)$ where (p) is the principal ideal generated by p.

(Consider the p^{th} cyclotomic polynomial and express

$$\prod_{r=1}^{p-1} (1-\rho^{r})$$

in two ways.)