Ph.D QUALIFYING EXAMINATION AUTUMN 1972-73

ALGEBRA

Parts III and IV

All questions in a given part have equal weight, although some are more difficult than others. Answer each question in a separate blue book. Write your name, the Part number, and the Question number on the cover of each blue book.

Notation: $\mathbb{Z}$ = integers; $\mathbb{Z}_n$ = cyclic group or order n;

$\mathbb{R}$ = reals; $\mathbb{C}$ = complex numbers $\mathbb{Q}$ = rationals.

Part III: Fields (25%)

Answer any four questions, and no more than four.

1. What are the Galois groups over the rationals of the following polynomials:

   (a) $X^3 - X + 1$

   (b) $1 + X + X^2 + X^3 + X^4$

   Prove all your statements, in particular the ones about irreducibility; for this you may quote theorems.

2. Let f be an irreducible polynomial in an indeterminate X with coefficients in a field K, and let $f_1(X), f_2(X), \ldots$ be the irreducible factors of f in $N(X)$, where N is a normal extension field of K. Show that the Galois groups of $f_1(X), f_2(X), \ldots$ over N are isomorphic.

3. Let $K \subsetneq L \subset K(X)$ be fields, with X transcendental over K. Prove

$$[K(X):L] < \infty .$$

4. Let A denote the ring $\mathbb{Q}[X]/(X^3-1)$.

   (a) What are the possible ring homomorphisms $f: A \to \mathbb{C}$ such that $f(1) = 1$?

   (b) By using these homomorphisms to determine a suitable map $A \to \mathbb{Q}$, show that if each of two rational numbers can be written in the form $a^3 + b^3 + c^3 - 3abc$, with a,b,c rational, then so can their product. (Hint: "Norm")

5. Let ρ be a primitive cube root of unity. Let G denote the cyclic group $\mathbb{Z}/9\mathbb{Z}$. Write the group ring

$$\mathbb{Q}(\rho)[G]$$

as a direct sum of fields.

6. Show from first principles that if $L = K(\alpha)$, α algebraic, then $\exists$ only finitely many intermediary fields M with $K \subset M \subset L$. (Hint: If $X^r + b_{r-1}X^{r-1} + \cdots$ is the irreducible polynomial of α over M, consider $K(b_1,\ldots,b_{r-1}) \subset M$).

7. Let k be a field of characteristic p. Suppose $a \in k$ such that the polynomial $X^p - X - a = f(X)$ has no roots in k. Prove $f(X)$ is then irreducible over k, and that, if α is a root of $f(X)$, then $k(\alpha)$ is Galois over k, with cyclic Galois group. (Hint: $\alpha + 1$).

Part IV: Rings (25%)

Answer any <u>three</u>, and no more than three.

1. Describe the ring of algebraic integers in the field $\mathbb{Q}(\sqrt{3})$. Justify your conclusion.

2. Let k be a field, and $G = k[X_1,\ldots,X_n]$ a polynomial ring. As a k-vector space, we may view

$$G = \bigoplus_{n=0}^{\infty} G_n ,$$

where G_n is the space of homogeneous polynomials of degree n. Let $f \in G_d$, some d, and let I denote the principal ideal $(f)G$. If we write $I_n = I \cap G_n$, and $\overline{G} = G/I$, we can write, as above

$$\overline{G} = \bigoplus_{n=0}^{\infty} \overline{G}_n ,$$

with $\overline{b}_n = G_n/I_n$. Let $H\colon \mathbb{N} \to \mathbb{N}$ be the function $H(n) = \dim_k \overline{G}_n$. Show that there is a polynomial P in one variable with rational coefficients such that for all n sufficiently large, $H(n) = P(n)$. (<u>Hint</u>: binomial coefficients)

3. Let A be a commutative ring with 1. An A-module P is called <u>projective</u> if, for any surjective A-module homomorphism $f\colon X \to Y$, and any A-homomorphism $g\colon P \to Y$, there exists an A-homomorphism $h\colon P \to X$ with $g = f \circ h$. Show that P is projective if and only if it is a direct summand of a <u>free</u> A-module.

4. Let F be a finite extension field of a field K. Show that if F is separable over K, then for any extension field E of K $F \otimes_K E$ is isomorphic to a direct sum of fields.

5. Let G be a group of order n, and let k be a field with $\text{char}(k) \nmid n$. Show that the group algebra $k[G]$ is semi-simple.

6. Give an example of a commutative noetherian ring R with the property: For any integer n, there is an ideal of R which cannot be generated by fewer than n-elements.