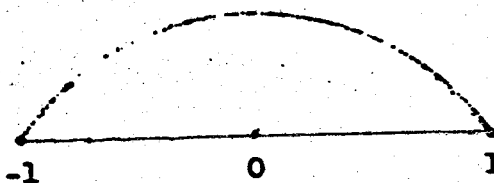


COMPLEX ANALYSIS

PART I

Do six problems. All problems have equal weight. Use a separate blue book for each problem. Write your name, Part I, and the number of the problem on the front of the blue book.

1. Let $u(z)$ be a harmonic function in the circular segment shown below.



Assume that $u = 1$ on the circular arc and $u = 0$ on the straight segment. Determine the locus of points at which $u = \frac{1}{2}$.

2. Let $f(z)$ be analytic in the punctured disk $0 < |z| < 1$. Assume that $f'(z)$ is square integrable in the unit disk. Prove that 0 is a removable singularity of $f(z)$.

3. Let a and b be unequal complex numbers. Show that every circle passing through a and b is orthogonal to the circle $(1-\lambda^2)|z|^2 - 2\operatorname{Re}(\bar{z}a - \lambda^2\bar{z}b) + |a|^2 - \lambda^2|b|^2 = 0$, where $\lambda \neq 1$ is a real number.

4. Let $p(z) = z + a_2z^2 + \dots + a_nz^n$ be univalent in $|z| < 1$. Show that $n|a_n| \leq 1$.

Hint: Consider the zeros of $p'(z)$.

5. Find a conformal map of the region bounded by the circles $C_0 = \{z \mid |z| = 1\}$ and $C_1 = \{z \mid |z - \frac{1}{2}| = \frac{1}{2\sqrt{2}}\}$ onto an annulus.

6. Let $f(z)$ be an entire function which is real on the real axis and purely imaginary on the imaginary axis. Prove that $f(z)$ assumes every finite value.

7. Let $f(z)$ be analytic in the unit disk. Assume that there exists a constant $c_1 > 0$ such that $|f'(z)| \leq c_1 |f(z)|$ for all z . Show that there exists a positive constant c_2 , depending only on c_1 , such that $|f(z_1)| \leq c_2 |f(z_2)|$ for all z_1, z_2 with $|z_1| < 1, |z_2| < 1$.

8. Show that $\int_0^{\infty} \frac{\cos ax}{\sqrt{x}} dx = \int_0^{\infty} \frac{\sin ax}{\sqrt{x}} dx = \frac{1}{2} \sqrt{\frac{2\pi}{a}}, \quad a > 0.$