

REAL ANALYSIS

Part I

Work any seven of the following problems. Where appropriate, state whether true or false and justify your answer by proofs, counterexamples, outline of proofs, pictures, etc., giving as much detail as time allows.

Write the answers to problems 1-5 in one blue book and those to problems 6-11 in another. On each blue book cover write your name and the numbers of the problems you have chosen.

$\mathbb{R}$  is the real line,  $\mathbb{R}^2$  the plane, etc.

1. Let  $\{f_n\}$  be a sequence of continuous positive functions on  $[0,1]$ .

Suppose that

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = 0.$$

Then there exists a subsequence of the  $\{f_n\}$  which converges to 0 except perhaps at a countable subset of  $[0,1]$ . True or false? Justify your answer.

2. Let  $\mu$  be a finite-valued Borel measure on  $\mathbb{R}$ ,  $f$  a real-valued continuous function on  $\mathbb{R}$  satisfying

$$\lim_{|x| \rightarrow \infty} f(x) = 0.$$

Evaluate

$$\lim_{t \rightarrow \infty} \int_{-\infty}^{\infty} f(tx) d\mu(x)$$

and

$$\lim_{t \rightarrow 0} \int_{-\infty}^{\infty} f(tx) d\mu(x).$$

Justify your answer.

3. Let  $E$  be a measurable subset of  $\mathbb{R}$ . Suppose that  $0$  does not belong to  $E$ . Show that the set

$$\{(x, 1/x); x \text{ in } E\} \subset \mathbb{R}^2$$

has  $\mathbb{R}^2$ -measure zero.

4. Let  $f$  be a real-valued continuously differentiable function on  $\mathbb{R}$  which is in  $L_2(\mathbb{R})$ ; that is

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty.$$

Prove that there is a sequence  $\{x_n\}$  tending to infinity such that

$$\lim_{n \rightarrow \infty} f(x_n) f'(x_n) = 0.$$

5. Show for  $0 < \alpha < 1$ , that the function

$$\varphi(x) = \sum_{n=1}^{\infty} \frac{1}{2^{n\alpha}} e^{i2^n x}$$

belongs to the function class  $\text{Lip } \alpha$ ; that is, there exists a constant  $c$  such that

$$|\varphi(x+h) - \varphi(x)| \leq c h^\alpha$$

for all  $x$  and  $h$ .

6. Construct a subset  $A$  of  $[0,1]$  such that: (a)  $A$  is of measure zero and (b)  $[0,1] \subset A+A \equiv \{x+y; x,y \text{ in } A\}$ . Prove that your constructed set satisfies properties (a) and (b). (Hint: Try a Cantor type set)

7. Let  $\{x_n\}$  be a sequence of points in  $[0,1]$  such that  $x_n \rightarrow 0$ . There exists a non-negative  $C^\infty$  function on  $[0,1]$  with  $f(x_n) = 0$  for all  $n$  and  $f(x) > 0$  if  $x$  is not one of the  $x_n$ . True or false? Prove your assertion.

8. Show that if  $f$  is a function of bounded variation on  $[0,1]$  there is a constant  $C$  such that

$$\left| \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) - \int_0^1 f(x) dx \right| \leq \frac{C}{n} .$$

Is this true for all Riemann integrable functions? Justify your answer.

9. Suppose that  $k(x)$  is  $L_2$  on  $[-1,1]$  and define the integral operator:

$$Kf(x) = \int_0^1 k(x-y)f(y)dy .$$

Show for the set of functions  $f$  satisfying  $\int_0^1 |f(x)|^2 dx \leq 1$  that the functions  $Kf(x)$  are equicontinuous for  $x$  in  $[0,1]$ . Does  $K$  define a compact linear operator on  $L_2(0,1)$ ? Justify your answer.

10. Let  $X$  be an infinite dimensional Banach space. Show that there exists a linear functional on  $X$  which is not bounded.

11. Let  $X$  be an infinite dimensional Banach space and let  $V$  be a weakly open subset of  $X$ . Show that  $V$  is unbounded.