Algebra Qualifying Exam September 19, 1979

1. For a group G, recall the commutator subgroup, is the subgroup G' of G generated by all commutators $[g,h] = g^{-1}h^{-1}gh$.

   (a) Show that G' is a normal subgroup of G and the quotient group G/G' is Abelian.

   (b) Prove if N is a normal subgroup of G, and G/N is Abelian, then N contains G'.

2. (a) Let p be a prime and n a positive integer. How many (non-isomorphic) abelian groups of order p^n are there?
   Describe them explicitly in the case p^3.

   (b) If $p_1, p_2, \cdots p_s$ are distinct primes how many abelian groups of order $p^{n_1} p^{n_2} \cdots p_s^{n_s}$ are there?
   In particular, describe all abelian groups of order 200.

3. A set X of generators of a group G is called minimal if G is not generated by a proper subset of X.

   (a) Describe two different minimal sets of generators for the multiplicative group of the non-zero rational numbers Q^x.

   (b) Describe an automorphism of Q^x which is different from $\sigma(a) = \pm a$.

4. Let V be a 3-dimensional vector space over the complex numbers. Let $\{e_1, e_2, e_3\}$ be a basis in V. Define the linear mapping $F: V \to V$ by

$$Fe_1 = e_1 + e_3$$

$$Fe_2 = e_2 + e_3$$

$$Fe_3 = e_2 - e_3$$

(a) What is the matrix representing F with respect to this basis?

(b) Compute Trace and Determinant of F.

(c) Find the eigenvalues and corresponding eigenvectors of F.

(d) What is the Jordan normal form of F?

5. A linear transformation G on an n dimensional vector space V is said to be <u>cyclic</u> if there is some vector V such that $v, Tv, T^2v, \ldots, T^{n-1}v$ is a base for V.

(a) Suppose T is irreducible on V (that is T leaves no non-trivial subspace of V invariant). Prove T is cyclic.

(b) Suppose there exists a base for V of eigenvectors of T. Prove T is cyclic if and only if the eigenvalues of T are distinct.

6. Let V be a finite dimensional Euclidean vector space over the reals. Let (u,v) for $u,v \in V$ denote the positive definite inner product in V.

   (a) What is meant by an ortho-normal basis of V?

   (b) Sketch an algorithm for constructing an O.N. basis of V.

   (c) Let $F: V \to V$ be a symmetric linear map, i.e. $(Fu,v) = (u,Fv)$. Prove that there is an O.N. basis of V in which F is diagonal.

7. Find the degree over Q of the following.

   (a) $\sqrt[5]{3} + \sqrt[4]{2}$

   (b) $\sqrt[3]{2} - \sqrt[3]{7}$

8. Let F denote a field containing 16 elements.

   (a) Is F unique up to isomorphism? Briefly explain your answer.

   (b) What is the structure of the multiplicative group of F?

   (c) Describe all the subfields of F.

   (d) Describe all automorphisms of F and relate the corresponding fixed fields to the subfields of F.

9. Let R be a commutative ring with identity.

(a) Define "prime ideal" and "maximal ideal" in R and prove that every maximal ideal is prime.

(b) Give an example for R and a non-zero ideal J in R where J is prime but not maximal.

(c) Suppose $R = Q[x]$, the ring of polynomials with rational coefficients. Describe all the maximal and all the prime ideals of R.