PRELIMINARY EXAMINATION - ANALYSIS

September 22, 1982

1:30-4:30

1. ____________________

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7. ____________________

8. ____________________

TOTAL ________________

1. Using Lebesgue's monotone convergence theorem, prove Fatou's Lemma.

2. Let g be analytic on $\{z: |z| < 1\}$ and assume that $|g(z)| = |z|$ for all $|z| < 1$. Show that $g(z) = e^{i\theta}z$ for some constant $\theta \in [0,2\pi]$.

3. a) State the Hahn-Banach Theorem

b) Let T be a linear subspace of a normed linear space X. Let y be an element of X whose distance to T is at least δ, i.e.,

$\|y-t\| \geq \delta$ for all $t \in T$. Prove that there exists a bounded linear functional f on X with $\|f\| \leq 1$, $f(y) = \delta$ and such that $f(t) = 0$ for all $t \in T$.

4. Evaluate

a) $\int_0^\infty \frac{\cos x}{x^2+b^2}\,dx;\ b > 0$

b) $\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$

5.

a) Let $f \in \mathcal{L}_p(\mu)$ and let $\varepsilon > 0$. Show that

$$\mu(\{x \in X : |f(x)| > \varepsilon\}) \leq \varepsilon^{-p} \int |f|^p \, dx$$

b) Let $\{f_n\}$ be a sequence of some $\mathcal{L}_p(\mu)$, $1 \leq p < \infty$.

If $\lim_{n\to\infty} \| f_n - f \|_p = 0$ in $\mathcal{L}_p(\mu)$, then $\{f_n\}$ converges in measure to f.

c) Show by a counterexample that the converse of b) is false.

6. Suppose $f \in \mathcal{L}^1(R)$, $g \in \mathcal{L}^1(R)$. Prove that if

$$h(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy \equiv (f*g)(x), \text{ then } h \in \mathcal{L}^1(R) \text{ and}$$

$$\|h\|_1 \le \|f\|_1 \, \|g\|_1 \ .$$

7.

a) State the Baire Category Theorem

b) Prove the existence of an everywhere continuous, nowhere differentiable function.

8.

a) State Rouche's Theorem

b) Let $f|z| = \sum_{n=0}^{\infty} a_n z^n$. Assume $a_0 = 0$, $a_1 = 1$.

Prove that f is 1-1 in the unit disk $\{z: |z| < 1\}$ if

$\sum_{j=2}^{\infty} j|a_j| \leq 1$.