ANALYSIS PH.D. QUALIFYING EXAM

September 21, 1983

1) (a) State rigorously Fatou's lemma.

(b) Use (a) to prove that the L_p spaces are complete; $1 \le p \le \infty$.

2) (a) Prove:

Let X be a countably compact metric space. Then given $\varepsilon > 0$, there are a finite number of points $x_1, x_2, \ldots, x_n$ of X such that for all $x \in X$, there is an x_k such that

$$\rho(x, x_k) < \varepsilon$$

(b) Prove that a countably compact metric space is separable.

3) (a) Give an example of a convergent sequence which is uniformly bounded on a compact set and which does not contain a uniformly convergent subsequence.

(b) Give an example of a uniformly bounded sequence of continuous functions on a compact set for which there does not exist a subsequence converging pointwise on the set.

4) If $a > 0$, show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2}\, dx = \pi \frac{\sin a}{a}$$

5) Prove: If $f(z)$ has an essential singularity at $z = \alpha$ and if γ is any complex number, then in every neighborhood of α, $f(z)$ comes arbitrarily close to γ.

6) (a) Let $\{f_n\}$ be a sequence of functions in L_p; $1 < p < \infty$, which converge almost everywhere to a function f in L_p. Suppose there is a constant M such that $\| f_n \| \le M$ for all n. Then for each function g in L_q,

$$\int fg = \lim \int f_n g .$$

(b) Is the result true for $p = 1$?

7) Show that $e^z = 2z+1$ has exactly one root in $|z| < 1$.

8) Suppose $f \in L_1(R)$, $g \in L_1(R)$. Prove that if

$h(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$, then $h \in L_1(R)$ and

$$\| h \|_1 \le \| f \|_1 \; \| g \|_1$$