ALGEBRA PRELIMINARY EXAM

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(1) V is a 3 dimensional vector space over the real numbers and the three vectors $\{e_1, e_2, e_3\}$ are a basis of V. L is a linear mapping of V into itself defined by the action

$$Le_1 = e_1 + e_2 \qquad Le_2 = 2e_2 + e_3 \qquad Le_3 = 5e_2 - 2e_3$$

Determine the eigenvalues and corresponding eigenvectors of L and then give a new basis (expressed in terms of the e's) in which L is represented by a diagonal matrix.

(2) Let V be as described in question (1) and let (a,b) denote a positive definite symmetric inner product on V (a,b in V). Describe what is meant by an ortho-normal basis of V with respect to this inner product. If S is a symmetric linear map of V into itself, so that (Sa,b) = (a,Sb) for all a,b in V, then give a proof that there is an ortho-normal basis of V in which S is represented by a diagonal matrix.

(3) Answer true or false.

(a) If two real matrices have the same characteristic polynomial then they are similar over the real numbers?

(b) Every subspace U of a finite dimensional real vector space has a complementary subspace? (i.e. a subspace which is disjoint from U and whose sum with U is the whole space).

(c) If N is a nilpotent real matrix (i.e. some power of N is 0) then all eigenvalues of N are 0 ?

(d) If L is a linear mapping of a real vector space into itself then every subspace which is invariant under the action of L has a complementary subspace which is also invariant under L ?

(e) If U and W are subspaces of a real vector space then the relation $\dim(U) + \dim(W) = \dim(U \wedge W) + \dim(U+W)$ holds?

(4) Up to isomorphism, how many different fields contain exactly 64 elements? For each such field F, describe the structure of the additive and multiplicative groups of F. Describe the group of automorphisms of F and relate its subgroups to the subfields of F.

(5) Let z be one of the primitive 12-th roots of unity (i.e. $z^{12} = 1$ but no smaller positive power of z equals 1). Give the minimal polynomial for z and determine the degree of the field $Q(z)$ over the field Q of rational numbers.

(6) Answer True or False.

(a) The field of complex numbers is the algebraic closure of the rational field?

(b) There are exactly two automorphisms of the complex field (namely the identity and complex conjugation)?

(c) The only automorphism of the field of real numbers is the identity?

(d) If a polynomial with integer coefficients is irreducible over the integers then it is also irreducible over the rationals?

(e) If F is a finite field then there always exist irreducible polynomials over F of arbitrary degree?

(7) Let F be any field and $F[x]$ the ring of all polynomials with coefficients in F. Give a proof that every ideal in $F[x]$ is a principal ideal.

(8) An exercise on the number 496 (= 1984 ÷ 4): Start by factoring it correctly. Then determine how many abelian groups of order 496 exist (up to isomorphism, of course). Now use the Sylow theorems to show that any group of order 496 must always be solvable. Lastly, if possible, describe an example of a non-abelian group of order 496.

(9) M is a free abelian group written additively and with 3 generators (i.e. M is a module over the integers of dimension 3). Suppose $\{a_1, a_2, a_3\}$ are a basis of M and that N is the subgroup spanned by the elements $2a_1 - a_2$, $2a_2 - a_3$, $2a_3 - a_1$. Describe the structure of the quotient M/N.