Analysis Qualifying Exam

Fall 1985

Time Limit: 3 hours. Answer as many questions as possible.

1) Prove Egoroff's Theorem: If f_n, f are measurable functions on $[0,1]$ and if $f_n \to f$ a.e., then for each $\epsilon > 0$, there exists a set E_ϵ of Lebesgue measure $< \epsilon$ such that $f_n \to f$ uniformly on the complement of E_ϵ in $[0,1]$.

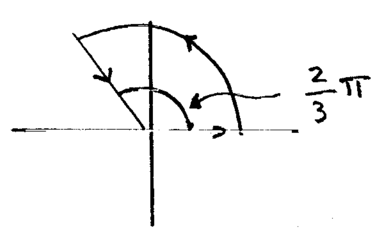
2) Consider $(R^2, <)$ where $<$ is the simple order relation on R^2 given by <u>dictionary</u> <u>or</u> <u>lexicographic</u> <u>order</u>.
   (a) Give the definition of dictionary order on R^2. That is, let $a \times b$, $c \times d \in R^2$. What is the definition of $a \times b < c \times d$. ((We write $a \times b$ rather than (a,b) to avoid notational difficulty - see part (b).))
   (b) Describe the open interval $(2\times 7, 5\times 3)$ in the order topology on $(R^2, <)$.

3) Let $R^\omega = \prod_{j=i}^{\infty} R_j$ where $R_j = R$ for each j. Let R^∞ be the subset of R^ω consisting of all sequences that are eventually zero. That is, all $(x_1, x_2, \ldots)$ such that $x_j \neq 0$ for only finitely many values of j. What is the closure of R^∞ in R^ω in the box and product topologies? [Recall that the box topology for $\prod_{\alpha \in \mathcal{J}} X_\alpha$ is generated by a basis consisting of all sets of the form $\prod_{\alpha \in \mathcal{J}} U_\alpha$ where U_α is open in X_α for each $\alpha \in \mathcal{J}$.] Explain.

4) Evaluate the integrals:

a) $\displaystyle\int_0^\infty \frac{dx}{1+x^3}$ b) $\displaystyle\int_0^\infty \frac{\cos x}{x^2+b^2}\,dx$

by means of the method of residues. Hint: For (a) integrate over:



5) Let the mapping $G: \mathbb{R}^4 \to \mathbb{R}^2$ be defined by the formula

$$G(x,y,u,v) = (xv+yu-1, xy-uv) .$$

Show that the implicit-function theorem implies that u and v are defined implicitly as a function of x and y, in a neighborhood of $(1,0)$ by the equation $G(x,y,u,v) = (0,0)$, and compute $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$, at $(1,0)$.

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6) State and prove both Liouville's theorem and the fundamental theorem of algebra.

7) A map f from an open set $U \subset \mathbb{R}^n$ into $\mathbb{R}^n$ is said to be an <u>isometry</u> if $\|f(x) - f(y)\| = \|x - y\|$.

(a) Show that any C^1 isometry preserves angles.

(b) Use this fact to conclude that if f is <u>orientation preserving</u> and U is an open subset of the complex plane then f must be of the form $f(z) = cx + d$, $|c| = 1$.

8) Write an essay on the theory of Fourier series. If possible, include a discussion of the Dirichlet kernel.

9) The Poisson kernel $P(r,x)$ is defined as $\sum_{n=-\infty}^{\infty} r^{|n|}e^{inx}$ for $0 \leq r < 1$ and is equal to $\dfrac{1 - r^2}{1 - 2r\cos x + x^2}$.

For each $f \in L_1(0,2\pi)$ and each $r < 1$ define

$$f_r(x) = \frac{1}{2\pi}\int_0^{2\pi} P(r,x-y)f(y)\,dy .$$

Prove that $\|f_r\|_1 \leq \|f\|_1$ for each r. Justify all steps in your proof.

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10) Let D be the open unit disc in the complex plane. The functions $\frac{1}{z} + z$ and $i(\frac{1}{z} - z)$ are holomorphic on $D - \{0\}$ and real on ∂D. Using this fact compute the dimension over the reals of the space of holomorphic functions on $D - \{0\}$, continuous on $\partial D \cup [D - \{0\}]$, and which are either analytic on D or have a pole of order one at $\{0\}$.

11) Assume r, s, t are real numbers such that $r \geq 1$, $s \geq 1$, $\frac{1}{r} + \frac{1}{s} = \frac{1}{t}$. Prove that in any measure space $f \in L_r$ and $g \in L_s \Rightarrow fg \in L_t$.