ALGEBRA QUALIFYING EXAM

September 29, 1986

1. (a) Find all the eigenvalues, and for each one, give a corresponding eigenvector of the matrix M.

   (b) Explain why there is, or is not, a matrix P such that PMP^{-1} is diagonal.

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

2. Let A be the free (additive) abelian group on 2 generators α, β. Let B be its subgroup spanned by the two elements $(10\alpha+16\beta)$, and $(6\alpha+12\beta)$. Describe the structure (as a finitely generated abelian group) of B and also of the quotient A/B.

3. Consider the cyclic group Z_{210} (of order 210)

   (a) List the number of elements of each order in Z_{210}.

   (b) Describe the structure of the automorphism group of Z_{210}.

4. If G is a non-abelian group let $z(G)$ denote its center, G' its commutator subgroup, and $\mathrm{Aut}(G)$ its automorphism group.

   (a) Show that there is a natural isomorphism of $G/z(G)$ into $\mathrm{Aut}(G)$.

   (b) Show that if $G = G'$ then G is not solvable.

5. Over the field $GF(2)$ describe the splitting field and resulting Galois group for the following polynomials:

   (i) $x^3 + 1$

   (ii) $x^5 + x + 1$

   (iii) $x^{16} + 1$

6. Let $\theta = \sqrt{-2}$ and consider the ring $Z[\theta]$.

   (a) Describe all the units in $Z[\theta]$.

   (b) Factor into prime ideals the ideal generated by 3.

   (c) Discuss the question of whether all ideals in $Z[\theta]$ are principal or not.

7. Prove that the field of real numbers has only the identity automorphism.

Please use a separate sheet for your answer to each of the 7 questions.