Analysis Qualifying Exam
September 30, 1987

Work as many problems as you can.

1. Let $\{X_n\}_{\alpha \in A}$ be a family of topological spaces. Define the product topology on ΠX_α and prove that for all α_o the mapping from ΠX_α to X_{α_o} given by $(x_\alpha) \to x_{\alpha_o}$ is continuous.

2. State and prove some form of the Baire category theorem.

3. Show that the set of all $\{a_n\} \in l^2(\mathbb{Z}^+)$ such that $|a_n| \leq n^{-1}$ for all n is a compact subset of $l^2(\mathbb{Z}^+)$ in the norm topology.

4. Show how to construct for each $\epsilon > 0$ a closed nowhere dense subset of $[0,1]$ with Lebesgue measure $> 1 - \epsilon$.

5. Suppose E_n are sets in a measure space and the measure of E_n is $\leq n^{-2}$. Show that almost every point belongs to at most finitely many of the sets.

6. State some form of the Radon-Nikodym theorem and give an example to show that the conclusion may be false if the hypotheses are not satisfied.

7. Construct a conformal mapping from the disc $D = \{z : |z| < 1\}$ to the strip $S = \{z : 0 < \text{Re} z < 1\}$.

8. (a) State the open mapping theorem for analytic functions.

   (b) State some version of the maximum principle for harmonic functions.

   (c) Deduce the maximum principle from the open mapping theorem

9. Let u be a function harmonic on $D = \{z : |z| < 1\}$ and continuous on $\overline{D}$.

   (a) Write down the Poisson integral formula for $u(re^{i\theta})$.

   (b) Suppose in addition that $u(z) > 0$ for all $z \in D$ and that $u(0) = 1$. Prove that $u(\frac{1}{2}) \leq 3$.

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10. Deduce the fundamental theorem of algebra from Liouville's theorem.

11. Let f be an entire function such that $2f(z) = f(2z)$. Prove that f is linear.

12. Suppose f is holomorphic in $D = \{z : |z| < 1\}$ and for each $z \in D$ there is some k such that $f^{(k)}(z) = 0$. Prove that f is a polynomial.