Algebra Preliminary Exam

Fall 1988

1. Let p be a prime in **Z**. Set $R = \{m/n \in \mathbf{Q} \mid p \nmid n\}$ and $J_i = \{m/n \in \mathbf{Q} \mid p^i \mid m, p \nmid n\}$. R is a subring of **Q** and the J_i are ideals of R.

   (a) Prove every element $x \in R - J_1$ is a unit in R.

   (b) Prove that J_1 is the unique maximal ideal of R.

   (c) Prove that $\{J_i : i \geq 1\} \cup \{R\} \cup \{0\}$ is the entire collection of ideals of R.

2. Recall that a module M for a ring R is <u>simple</u> or <u>irreducible</u> if the only submodules of M are {0} and M itself. Prove that if M is an irreducible R-module, then $\mathrm{Hom}_R(M,M)$, the ring of R-homomorphisms from M to M, is a division ring.

3. Let G be a non-solvable group of order 60.

   (a) Prove G has 6 five-Sylow subgroups and 24 elements of order five.

   (b) Prove G has 10 three-Sylow subgroups and 20 elements of order three.

4. Let G be a finite group with k conjugary classes. Determine the cardinality of the set

$$\{(g,h) \mid gh = hg\}$$

in terms of $|G|$ and k.

5. Let $M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$, a 3×3 complex matrix.

   (a) Find the minimal polynomial of M.

(b) Find the characteristic polynomial of M.

(c) Is M diagonalizable over C? Explain why, or why not.

6. Determine the Galois group of $x^4 - 4x^2 + 5$ over $\mathbf{Q}$.

7. Let ε be a primitive 18th root of unity.

(a) Determine the minimal polynomial of ε.

(b) Determine the degree of ε over Q, $[Q(\varepsilon):Q]$.

(c) Determine the Galois group $Gal(Q(\varepsilon)/Q)$ as completely as possible.

8. Let A be the (additive) Abelian group with generators a,b, and c satisfying the relations

$$\begin{aligned} 2a + 4b \phantom{{}+ 8c} &= 0 \\ 2a \phantom{{}+ 4b} + 8c &= 0 \\ 4a + 12b + 4c &= 0 \end{aligned}$$

(a) What is the rank of a maximal free Abelian subgroup of A?

(b) What is the structure of the torsion group of A?