Analysis Preliminary Examination
Fall 1989

This is a closed book examination.

You may use a calculator if you wish.

This examination is for a period of 3 hours. Use your time efficiently.

Write neatly. Leave margins. You may use or quote any theorems that you are not explicitly asked to prove.

Write only on 1 side of the paper.

Start each problem on a new sheet of paper.

Write your name at the top of each page..

1. State some form of the Radon-Nikodym theorem and give an example to show that the conclusion may be false if the hypotheses are not satisfied.

2. a) Define what is meant by a complete metric space.

b) Let (X,d) be a complete metric space and let $T : X \to X$ be a contraction. State what it means to be a contraction. Prove that T has a unique fixed point.

3. Suppose $f:[a,b] \to R$ is nondecreasing and that $g:[a,b] \to R$ is continuous. Show that there exists $c \in [a,b]$ such that the Riemann-Stieltjes integral $\int_a^b g\, df$ equals $g(c)\,(f(b) - f(a))$.

4. a) What does it mean to say that $f:[a,b] \to C$ is of bounded variation (BV) ?

b) What does it mean to say that $f:[a,b] \to C$ is absolutely continuous (AC) ?

c) Give an example (and prove that it is one) of a function $f \in BV$ but $f \notin AC$.

5. Let f be an entire function. Suppose $f(3z) = 3f(z)$ for all $z \in C$. Prove that f is linear.

6. Evaluate the integral $\int_\gamma \frac{e^z(z+1)}{(z-1)(z-3)}\,dz$ where $\gamma : [0,1] \to R$ is given by $\gamma(t) = 2e^{i2\pi t}$.

7. a) State some version of the Baire category theorem.

b) Let Q = rationals. Show that $Q \times Q \subset R^2$ is not the intersection of countably many open sets in R^2.

8. Let $0 < a < 1$. By integrating $\frac{1}{(z-a)(z-a^{-1})}$ around the unit circle show that $\int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2} = \frac{2\pi}{1-a^2}$.

9. Evaluate the following limit: $\lim_{n\to\infty} \int_0^n (1-\frac{x}{n})^n e^{x/2}\,dx$. Justify your computation.

10. Let $U = \{z \in C : |z| < 1\}$. Suppose $f: U \to U$ is holomorphic, $f(0) = 0$, and $|f'(0)| = 1$. Show that $f(z) = \alpha z$ for some $\alpha \in C$ with $|\alpha| = 1$.

11. Let $P_n(z) = 1 + z + \frac{z^2}{2!} + \dots + \frac{z^n}{n!}$ and define $r_n = \min\{|\zeta| : P_n(\zeta) = 0\}$. Prove that $\lim_{n\to\infty} r_n = \infty$.

12. a) State and prove Fatou's lemma.

b) Prove that the L^p spaces are complete for $1 < p < \infty$.

c) What can you say about L^1 and L^∞? Are they complete?