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Analysis Preliminary Examination
September 27, 1990

Instructions: Work as many problems as you can. Think and do not panic! You are not expected to do all problems.

1. Consider the function $f(x,y) = \begin{cases} (0,0) \text{ if } (x,y) = (0,0) \\ \dfrac{(x+y)^2(x-y)}{x^2+y^2} \end{cases}$

   (a) Compute $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$.

   (b) Explain why $f(x,y) = 0$ does not determine y as a single valued function of x near 0.

2. Show that the unit disc is not conformally equivalent to the whole complex plane.

3. Let f be a non-zero holomorphic function on the unit disc D such that $\log |f|$ is constant on ∂D. Show that f is constant.

4. Evaluate the following integrals:

   (a) $\displaystyle\int_{S^1} \frac{\cos 3z}{z^2}\,dz$ (b) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$

5. State:
   (a) The Dominated Convergence Theorem
   (b) The Monotone Convergence Theorem

6. (i) Define "Complete Metric Space", (ii) State the Baire Category Theorem, (iii) Let $P_n(x,y)$ be a sequence of polynomials (non-identically zero) in two variables. Prove that there is an (x_0,y_0) which is not a zero of any of the P_n.

7. (a) Give the definition of a Banach space.

(b) What does it mean for a Banach space to be seperable?

(c) The Banach spaces c_0 of sequences converging to 0, and l_∞ of bounded sequences, are well known. What are the norms on these spaces.

(d) Show that c_0 is seperable but that l_∞ is not.

8. Let X,Y be compact metric spaces with $f : X \to Y$ a 1:1 continuous onto map. Show f is a homeomorphism.

9. The rational fractions between 0 and 1 are arranged in order $\frac{1}{2}\ \frac{1}{3}\ \frac{2}{3}\ \frac{1}{4}\ \frac{2}{4}\ \frac{3}{4}\ \frac{1}{5}\ \frac{2}{5}\ \cdots\cdots$. Let P_n be the n^{th} fraction in this sequence. Show that $\sum_{n=1}^{\infty} (P_n)^{n^2} z^n$ converges for all z in the complex plane. Hint: Use the root test.

10. Let $\varphi : R \to R$ be a real valued function with $|\varphi(x) - \varphi(y)| < \frac{1}{2}|x - y|$. On the Banach space $E = C[0,1]$ of continuous functions on the unit interval consider the mapping $T_\varphi : E \to E$ given by $T_\varphi(f) = \varphi(f) - f$. Show that T_φ has a zero in E; i.e. $\exists\, f \ni : T_\varphi(f) = 0$.