Analysis Preliminary Examination **September 23, 1992**
1:00 - 4:00 p.m.

1. Prove that the closed unit ball in the normed vector space
$\ell^2 = \{(a_1, a_2, \ldots) : \|\{a_n\}\| = \left(\Sigma |a_n|^2\right)^{\frac{1}{2}} < \infty\}$ is not compact.

2. The set of $n \times n$ real matrices is given the topology of $\mathbb{R}^{n^2}$. Show that the set of matrices A such that $AA^t = I$ (t= transpose) is compact, but that the set of matrices A such that $\det A = 1$ is not compact.

3. A base for a certain topology $\mathfrak{I}$ in $\mathbb{R}^n$ consists of all finite intersections of sets of the form $p^{-1}(\mathbb{R}\setminus\{0\})$ where p is a polynomial in n variables. Show that $\mathfrak{I}$ defines a T_1 topology in $\mathbb{R}^n$ which is not metrizable.

4. Is the function $\begin{cases} x^2 \sin\frac{1}{x} & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{cases}$ of bounded variation on $[0,1]$?

   Is it absolutely continuous? Explain.

5. Prove that $\lim\limits_{n\to\infty} \int_0^\infty \frac{\cos x}{1+nx^2}\,dx = 0$.

6. Let E be a closed no where dense subset of $[0,1]$ with Lebesque measure $\frac{1}{2}$, χ_E its characteristic function. Does there exist a function $f \in C([0,1])$ such that

   a) $f = \chi_E$ except on a set of measure $< \frac{1}{4}$?

   b) $f = \chi_E$ a.e. ?

   Justify your answers.

7. Show that if E is a subset of $\mathbb{R}$ of finite Lebesque measure then $m(E \cap (E + x))$ is a continuous function of x . (Here m = Lebesque measure.)

8. Let $f(z) = z^2 - z - 12$ and put $g(z) = \frac{1}{f(z)}$. Find the first 3 terms of the power series expansion of g about $z = 0$. What is the radius of convergence of the series?

9. State and prove some version of the maximum modulus theorem.

10. Evaluate the integral $\int_0^\infty \frac{dx}{1+x^4}$.

11. (a) Suppose u is positive and harmonic in the open unit disc U and $\lim\limits_{r\to 1} u(re^{i\theta}) = 0$ for a.e. θ. Is it true that $u(z) = 0$ for all $z \in U$?

    (b) Prove that if f is analytic and nonzero in U then $\log |f(z)|$ is harmonic in U .