Analysis Preliminary Exam **September 22, 1993**

Please do 7 (out of 9) problems. Where appropriate, state whether true or false and justify your answer by proofs, counter-examples, outline of proofs, pictures, etc., giving as much detail as time allows.

1. f is a continuous function with $f(t+1) = f(t)$ and $f(t + \sqrt{2}) = f(t)$. Show that $f(t) \equiv f(0)$.

2. True or False. Suppose $y(x)$ satisfies a differential equation of the form $y' = p(y)$ where p is a polynomial. Then the domain of definition of y can be taken to be the entire real line.

3. Suppose f is a non-negative Lebesgue integrable function on the unit interval I. Show that $\int_I f = 0$ implies that $f = 0$ a.e.

4. Let L_2 be the space of Lebesgue square integrable functions on the unit interval. Give an example of a subset of L_2 which is closed and bounded, but not compact with respect to the strong topology.

5. Let X be a set with n points $\{p_1, \ldots, p_n\}$ with the discrete topology. Describe the topological vector space C(X) of real-valued continuous functions on X. Suppose $f_k \in C(X)$ is a sequence. What does it mean for $f_k \to f \in C(X)$?

6. $T(\lambda)$ is a continuous $n \times n$ matrix-valued function of the real parameter λ.
   - State precisely what this means.
   - Show that the spectrum of $T(\lambda)$ depends continuously on λ. Be careful to state precisely what this means. (Pay special attention to points λ where $T(\lambda)$ has multiple eigenvalues.)
   - Show that the map $\lambda \to m(\lambda)$, where $m(\lambda)$ is the minimal polynomial of $T(\lambda)$, is not continuous with respect to λ.

7. Evaluate the integral

$$I(t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-ixt}\,dx}{x^2 + 2i\lambda x - x_0^2}$$

where $x_0 > \lambda > 0$ and $t \neq 0$

8. Let $f(z) = \dfrac{P(z)}{Q(z)}$

where P and Q are polynomials with no common factors.

- Under what conditions on P and Q can f be used to define a function F from the Riemann sphere to itself?
- What is $F(\infty)$ in terms of deg P, deg Q and the leading coefficients of P and Q?
- Describe the set of all f such that F is invertible.

9. (a) Show that the function $z \to z + \dfrac{1}{z}$ is an analytic isomorphism of the region outside the unit circle onto the plane from which the segment [-2,2] has been deleted.

(b) What is the image of the unit circle under this mapping? (Hint: Use polar coordinates.)