

***Analysis Preliminary Examination***  
***September 28, 1994***

1. True or False. Provide a brief outline of a proof or a counter-example.
  - a) A linear subspace of a vector space is closed.
  - b) A closed bounded set in a complete metric space is compact.
  - c) Given a closed convex set  $C$  and a point  $p$  in a Banach space, there is a unique point in  $C$  that is closest to  $p$ .
  
2. Is the space  $W[a,b]$  of continuous functions from the interval  $[a,b]$  to  $\mathbb{C}$ , with the inner product  $\langle f, g \rangle = \int_a^b f(t)\overline{g(t)}dt$ , a Hilbert space?
  
3. Show that a linear functional  $F$  on a normed space is continuous if and only if  $\ker F$  is closed.
  
4. A metric space  $(V, \rho)$  is said to be totally bounded if for every  $\varepsilon > 0$  there is a finite set  $A_\varepsilon$  such that for every  $x \in V$  there is at least one point  $a \in A_\varepsilon$  satisfying  $\rho(x, a) \leq \varepsilon$ . Show that every totally bounded metric space is separable.
  
5. Let  $\mu$  be a positive measure on the real line with Fourier Transform  $c(k) = \int e^{ikx} d\mu(x)$ . Show that it is impossible to have  $c(0) = 1$  and  $c(1) = 2$ .
  
6. Give an example of a sequence  $f_n: [0,1] \rightarrow \mathbb{C}$  of continuous functions which are uniformly (in  $n$ ) bounded, converge weakly in  $L^2$  to a continuous function, but such that no subsequence converges strongly.
  
7. Let  $f$  be an entire function satisfying  $f(z + 2\pi) = f(z)$  and  $|f(z) - e^z| \leq \frac{1}{|y|}$  for  $|y| \geq 100$ , where  $z = x + iy$ . Show that  $f(z) = e^z$ .

8. Evaluate  $\int_0^{\infty} \frac{x^{-r} dx}{1 + 2x \cos \lambda + x^2}$  using residues, where  $-1 < r < 1$ , and  $-\pi < \lambda < \pi$ . Use the following type of contour:

