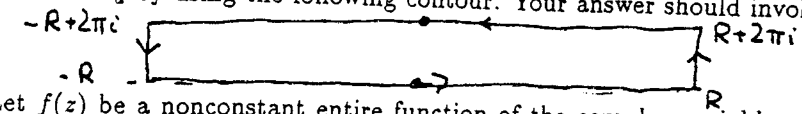
Analysis Preliminary Exam
Sept. 1995
1). Set

$$I_a = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}dx,$$

for $0 < a < 1$.

a) Show that $I_a$ converges.

b) Evaluate $I_a$ by using the following contour. Your answer should involve $sin(a\pi)$.




2) Let $f(z)$ be a nonconstant entire function of the complex variable $z$. Show that the image of $f$ is dense.

3) Find a one-to-one conformal map from the complement of the nonnegative real axis $[0,\infty)$ onto the unit disc.

4) Let $B$ be the unit ball in a Hilbert space. Show that $B$ is compact if and only if $H$ is finite-dimensional.

5) A real number $a$ is said to have type $\nu$, $\nu > 0$, if there exist infinitely many distinct rationals $p/q$ such that $|a - (p/q)| < q^{-\nu}$. Let $A_\nu$ denote the set of real numbers of type $\nu$.

a) Show that the complement of $A_\nu$ is a set of the first category.

b) Show that $A_\nu$ has Lebesque measure zero, provided $\nu > 2$.

HINTS: First consider the intersection of $A_\nu$ with the unit interval $I$. Express $A_\nu \cap I$ as a countable intersection of sets, each one of which is the countable union of a FINITE collection of intervals.

6). Give an example of an ordinary differential equation of the form $\frac{dy}{dt} = f(y)$ for which there are solutions $y(t)$ blows up (tend to infinity) in finite time $t$, and for which $f$ is a polynomial.

7). Let $S$ denote the set of all infinite sequences $\{s_n\}_{n=0}^{\infty}$ of complex numbers which solve the recursion relation

$$s_{n+2} = As_{n+1} + Bs_n.$$

a) Show that $S$ forms a vector space over the complex numbers.

b) Show that the dimension of $S$ is two by describing a basis.

c) For what parameter values, $A, B$ is it true that $\{s_n\} \in S$ implies that $lim_{n\to\infty} s_n = 0$? HINT: Try to find a basis for $S$ for which each sequence is of the form $s_n = z^n$.

8). Let $\{f_n\}$ be a sequence of differentiable functions on the unit interval $I$ which satisfy $f_n(0) = 0$ and $|\frac{df_n}{dt}(t)| \leq 1$

a) Show that the sequence admits a convergent subsequence.

b) Let $\{g_n\}$ denote the convergent subsequence of part a) and $f_*$ the function to which it converges. Show that $\int_I g_n^2 \to \int f_*^2$ as $n \to \infty$.

c) With the same notation as in b), show that $\int_I \frac{1}{t^{3/2}} g_n(t)dt$ converges. To what?