

1. Let  $f$  be a continuous map from a compact metric space  $X$  to a metric space  $Y$ . Prove that  $f$  is uniformly continuous.
2. Let  $\{X_\alpha : \alpha \in I\}$  be a family of topological spaces, where  $I$  is an index set.
  - (a) Define product topology on  $\times_{\alpha \in I} X_\alpha$ .
  - (b) Prove the product of two connected spaces is connected.
3. Let  $F$  be a complex analytic function on the open unit disc  $D$  which is continuous and real valued on the boundary  $\partial D$ . Show  $f$  is constant.
4. Let  $f$  be a biholomorphic map of the complex plane to itself (i.e., both  $f$  and  $f^{-1}$  holomorphic). Show  $f(z) = az + b$ .
5. Let  $m$  be the space of  $2 \times 2$  real symmetric matrices and  $f$  a real valued function on  $m$ . Define what it means for  $f$  to be continuous. Define what it means for  $f$  to be differentiable.

Let  $\lambda(A)$  be the function which assigns to a  $2 \times 2$  real symmetric matrix  $A$  its largest eigenvalue  $\lambda$ .

- (a) Is  $\lambda$  a continuous function of  $A$ ?
- (b) Differentiable?

6. Let  $A$  be a symmetric  $n \times n$  matrix. Prove that the function  $f(v) = \frac{\|Av\|}{\|v\|}$ ,  $v \in \mathfrak{R}^n$ ,  $v \neq 0$  has a minimum.

7. Give examples of
  - (a) a function which is Lebesgue integrable but not Riemann integrable,
  - (b) a function in  $L_1[0,1]$  but not in  $L_2[0,1]$ ,
  - (c) a dense subset of  $[0,1]$  whose measure is zero.

8. Define the Laplace transform of a function  $f \in L_1(0,\infty)$  by  $F(s) = \int_0^{\infty} e^{-sx} f(x) dx$ 
  - (a) Show  $f$  is continuous for  $\text{Re}(s) \geq 0$ .
  - (b) Show  $f$  is analytic in the region  $\text{Re}(s) > 0$ .

9. (a) Find an analytic function  $g(z)$  taking the unit disc  $|z| < 1$  onto itself and satisfying  $g\left(\frac{1}{2}\right) = 0$ .
- (b) Suppose  $f$  is analytic in the unit disc  $|z| < 1$  and  $|f(z)| \leq 1$  there. Suppose also that  $f\left(\frac{1}{2}\right) = 0$ . How large can  $\left|f\left(\frac{1}{3}\right)\right|$  be?