Analysis Preliminary Exam

December 12, 1997
12:00 - 5:00pm

1. Suppose μ is a finite measure and $f_n \to f$ in $L^3(\mu)$. Prove that $f_n \to f$ in $L^2(\mu)$.

2. Suppose $f \in L^1([a,b])$ with Lebesgue measure, and $\int_a^b f(x)g(x)dx = 0$ for all continous functions g. Prove that $f = 0$ almost everywhere.

3. Define a topology on $[0,\infty)$ with basis consisting of the following sets: (i) all (a,b) with $0 < a < b < \infty$;(ii) the set $\{0\}$. Prove that the function f defined by $f(x) = \sin(1/x)$ if $x \neq 0$ and $f(0) = 6$ is continous from $[0,\infty)$ with this topology to $\mathbf{R}$ with its standard topology.

4. Prove that if X is a compact metric space and $f: X \to X$ is such that $d(f(x),f(y)) < d(x,y)$ whenever $x \neq y$ then there exists an element $x \in X$ such that $f(x) = x$. [Hint: show that $\{d(x,f(x)): x \in X\}$ is a compact subset of $[0,\infty)$ and consider its smallest element.]

5. Let f(z) be a holomorphic function in an open connected set U containing the disc $|z| \leq 1$. Suppose that f takes only imaginary values on the circle $|z| = 1$. Show that f is constant in U. [Hint: Consider the function $e^{f(z)}$]

6. Let $f(z)$ be a holomorphic function in the whole plane and suppose that there is an integer n and two positive real numbers R and M such that $|f(z)| \leq M|z|^n$ for $|z| \underset{\geqslant}{\cancel{\leq}} R$. Show that $f(z)$ is a polynomial of degree at most n.

7. a) Let X be a closed subspace of a Banach space Y. Prove that the restriction $f \mapsto f|X$ is a continuous linear epimorphism $Y^* \to X^*$

   b) Give an example of a continous monomorphism $\varphi : X \to Y$ of Banach spaces so that the mapping $f \mapsto f \circ \varphi$ from Y^* to X^* is not onto.

8. Let X and Y be Banach spaces and $L : X \to Y$ a continous linear operator. Prove that L is compact if L^* is compact. [Hint: A is compact $\Rightarrow$ A^* is compact]