Algebra Preliminary Examination, Fall 1998

Notation: Z = integers, Q = rational numbers, R = real numbers, C = complex numbers, for a finite group $G, o(G)$ denotes the order of G.

1. Let G be a $p-$group with $o(G) = p^n$ for some $n > 0$.
(a) Prove that the center of G is non-trivial, that is, $o(Z(G)) > 1$.
(b) Prove that any subgroup of order p^{n-1} of $G$is normal.

2. If G is a group of order 231 show that the 11-Sylow of G is in the center of G.

3. (a) State the definition of greatest common divisor in a commutative ring.
(b) Let R be a PID. Prove that any two elements a, b, not both zero, have a greatest common divisor d and that d can be written in the form

$$d = ax + by$$

for some $x, y \in R$.

4. Compute the Galois groups of the following polynomials over Q:
a) $x^3 - 1$ b) $x^4 + x^3 + x^2 + x + 1$ c) $(x^2 - 3)(x^2 - 2)$

5. Find the Jordan canonical form of the following matrix over the complex numbers:

$$A = \begin{pmatrix} 2 & 0 & 0 & 2 \\ -1 & 2 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

6. Show that the dimension of a splitting field E/F of a polynomial $f(x)$ of degree n is at most $n!$.

7. Let R be a commutative ring and let M, N be left $R-$modules.
An $R-$multilinear map

$$\alpha : M^n \to N$$

is called **alternating** if

$$\alpha(m_1, \ldots, m_n) = 0$$

whenever $m_1, \ldots, m_n \in M$ and $m_i = m_j$ for some $i \neq j$.
Prove that there is a natural alternating $R-$linear map

$$\nu : M^n \to \wedge^n(M)$$

which has the universal mapping property with respect to $R-$multilinear alternating maps $\alpha : M^n \to N$.

8. Let $\mathrm{Z}[i] = \{x + iy : x, y \in \mathrm{Z}\}$ be the ring of Gaussian integers (here $i = \sqrt{-1}$). Determine what the ring

$$\mathrm{R} \otimes_{\mathrm{Z}} \mathrm{Z}[i]$$

is isomorphic to. Include a proof.

9. Let V be a finite dimensional vector space over a field K. Let (,) be a non-degernate orthogonal inner product on V and assume that V contains a non-zero isotropic vector (i.e. a vector $0 \neq v \in V$ such that $(v, v) = 0$). Prove that V has a basis consisting of isotropic vectors.