

## Preliminary Exam, Algebra

September 22, 1999

1. Let  $p$  be a prime,  $n \in \mathbb{N}$ , and  $G$  a group of order  $p^n$ . Show that every subgroup of index  $p$  in  $G$  is normal in  $G$ .

2. An element  $e$  of a ring  $R$  is called an *idempotent*, if  $e^2 = e$ .

(a) Let  $p$  be a prime and  $n \in \mathbb{N}$ . Show that 0 and 1 are the only idempotents in  $\mathbb{Z}/p^n\mathbb{Z}$ .

(b) Determine all idempotents of  $\mathbb{Z}/1000\mathbb{Z}$ .

3. Consider the subset

$$U := \{(w, x, y, z) \in \mathbb{Z}^4 \mid x - 2y + 6z = 0 \text{ and } z \in 5\mathbb{Z}\} \subseteq \mathbb{Z}^4.$$

(a) Show that  $U$  is a subgroup of  $\mathbb{Z}^4$ .

(b) Find a  $\mathbb{Z}$ -basis of  $U$ .

(c) Determine the isomorphism type of  $\mathbb{Z}^4/U$ .

4. Find an element  $\alpha \in \mathbb{C}$  such that  $\mathbb{Q}(\alpha)/\mathbb{Q}$  is a Galois extension of degree 5.

5. Let  $A$  be an orthogonal operator acting on the 5-dimensional Euclidean space  $E^5 = \mathbb{R}^5$ . Suppose that the characteristic polynomial of  $A$  has two distinct roots  $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$  such that  $\bar{\lambda} \neq \mu$ . Show that there exist two orthogonal 2-dimensional subspaces  $E_1$  and  $E_2$  of  $E$  such that  $A(E_i) = E_i$ , for  $i = 1, 2$ .

6. Let  $V$  be a vector space of dimension  $n \geq 2$  and let  $u \in \Lambda^2 V$ . Show that  $u$  can be written as  $e \wedge f$  for some  $e, f \in V$  if and only if  $u \wedge u = 0$  in  $\Lambda^4 V$ . (Hint: Use induction on  $n$  and use a basis  $\{e_1, \dots, e_n\}$  of  $V$ . For the induction step  $n \rightarrow n + 1$ , write  $u = e_{n+1} \wedge v + w$  with suitable  $v \in V$  and  $w \in V \wedge V$ . Show that  $v \wedge w = 0$  and  $w \wedge w = 0$  and apply the induction hypothesis to  $w$ .)