Algebra Preliminary Exam

Fall 2002

1. Let G be a nonempty set with an associative product. Assume that (1) G contains a right identity 1_r, that is, $a1_r = a$ for all $a \in G$, (2) every $a \in G$ has a right inverse b : $ab = 1_r$. Prove that G is a group.

2. Prove that any non-abelian group of order 6 is isomorphic to S_3.

3. (a) State the definition of Euclidean domain.

(b) Let D be a Euclidean domain with function δ. Prove that $a \in D$ is a unit if and only if $\delta(a) = \delta(1)$.

4. Prove that any finitely generated module M over a PID D is a direct sum of a torsion module and a free module.

5. Let $a, b \in \mathbb{C}$ be algebraic over $\mathbb{Q}$. Prove that both $a + b$ and ab are algebraic over $\mathbb{Q}$.

6. Assume that $x^p - a \in \mathbb{Q}[x]$ is irreducible (here p is a prime). Determine the Galois group of $x^p - a$ over $\mathbb{Q}$.

7. Let $f : V \times V \to K$ be a symplectic form on K-vector space V.

(a) Show that there exists a K-linear map

$$\psi : \wedge^4(V) \to \wedge^2(V)$$

with $\psi(v_1 \wedge v_2 \wedge v_3 \wedge v_4) = f(v_1, v_2)v_3 \wedge v_4$ for all $v_1, v_2, v_3, v_4 \in V$.

(b) The product in the exterior algebra $\wedge(V)$ induces a bilinear map $\beta : \wedge^2(V) \times \wedge^2(V) \to \wedge^4(V)$. Composing these maps one obtains a map

$$\psi\beta : \wedge^2(V) \times \wedge^2(V) \to \wedge^2(V)$$

which defines an algebra structure on $\wedge^2(V)$. Prove that $\wedge^2(V)$ is not in general associative.

8. Let V and W be vector spaces over a field K. Show that $S(V \oplus W)$ and $S(V) \otimes_K S(W)$ are isomorphic graded K-algebras.