

Preliminary Exam, Algebra, Fall 2004

1. Let N be a non-trivial normal subgroup of a finite p -group G . Show that $Z(G) \cap N \neq 1$.

2. Let A, B be finite subgroups of group G . Prove that

$$|AB| = o(A)o(B)/o(A \cap B).$$

3. Let D be a finite domain without assuming the existence of 1. Prove that D is a field.

4. Let M be a finitely generated torsion module over a PID D . Let $I = \{a \in D \mid aM = 0\}$.

(a) Prove that I is an ideal of D .

(b) If $M = Dz_1 \oplus Dz_2 \oplus \cdots \oplus Dz_s$ such that $\text{ann}(z_i) = (d_i) \neq D$ and $d_i \mid d_{i+1}$ for $i = 1, \dots, s-1$ then $I = (d_s)$.

5. (a) Show that $x^5 - 6x + 3$ has exactly 2 non-real roots. (b) Determine the Galois group of $x^5 - 6x + 3$ over \mathbb{Q} .

6. Let ζ be a primitive 2004-th root of unity in the complex numbers and let K be a subfield of $\mathbb{Q}(\zeta)$.

(a) Show that K/\mathbb{Q} is a Galois extension.

(b) Let G be the Galois group of K/\mathbb{Q} . Show that G cannot have an element of order p for any prime in the interval $[3, 79]$.

(c) How many elements of order 2 can G have (when K varies)?

7. Let V be a vector space over a field K and let $T(V)$, $S(V)$, and $\Lambda(V)$ denote the tensor algebra, symmetric algebra, and exterior algebra of V , respectively. Let $f \in \text{Hom}_K(V, V)$ be invertible. Show that f can be uniquely extended to K -algebra automorphisms of $T(V)$, $S(V)$, and $\Lambda(V)$.

8. Let R be a commutative ring, let V be a free R -module of rank n , and denote by $V^* := \text{Hom}_R(V, R)$ its dual.

(a) Show that the map $\varphi: V^* \otimes_R V \rightarrow \text{Hom}_R(V, V)$, given by $(\varphi(f \otimes v))(v') := f(v')v$ for $f \in V^*$ and $v, v' \in V$ is an isomorphism of R -modules.

(b) Let v_1, \dots, v_n be an R -basis of V and let $f_1, \dots, f_n \in V^*$ denote its dual basis. Show that the element $\alpha := \sum_{i=1}^n f_i \otimes v_i \in V^* \otimes_R V$ is independent of the choice of the basis v_1, \dots, v_n .

9. Let A be a $n \times n$ real symmetric matrix and I_n be the $n \times n$ identity matrix. Prove that $tI_n + A$ is positive definite if t is very large.