Preliminary Exam in Algebra, Fall 2006

1. Let G be a finite group and let H and K be subgroups of G. Define the relation "$\sim$" on G by $g_1 \sim g_2$ if and only if there exist $h \in H$ and $k \in K$ such that $g_2 = hg_1k$.

(a) Show that "$\sim$" is an equivalence relation and that the equivalence class of $g \in G$ is equal to HgK. The equivalence classes are called *double cosets* and are denoted by $H\backslash G/K$

(b) Show that every double coset has a cardinality divisible by $\mathrm{lcm}(|H|, |K|)$.

(c) Let $g \in G$. Show that $|H| = |HgH|$ if and only if $g \in N_G(H)$.

(d) Assume that $g_1, \ldots, g_n$ are representatives of the equivalence classes $H\backslash G/K$. Show that $g_1^{-1}, \ldots, g_n^{-1}$ are representatives of $K\backslash G/H$.

2. (a) Define what it means that a ring R is a principal ideal domain.

(b) Let R be a subring of the ring S and assume that both are principal ideal domains. Let $a, b \in R$ and let $r \in R$ be a greatest common divisor of a and b. Moreover, let $s \in S$ be a greatest common divisor of a and b considered as elements of S. Show that there exists a unit u of S such that $r = us$.

3. Let R and S be commutative rings and let $f\colon R \to S$ be a unitary ring homomorphism.

(a) Show that if P is a prime ideal of S, then $f^{-1}(P)$ is a prime ideal of R.

(b) Find an example of $f\colon R \to S$ and a maximal ideal M of S such that $f^{-1}(M)$ is not a maximal ideal of R.

(c) Show that if f is surjective, then $f^{-1}(M)$ is a maximal ideal of R for every maximal ideal M of S.

4. Let V be a finite dimensional vector space over a field $\mathbb{F}$.

(a) Define what is meant by a non-degenerate alternating from $f\colon V \times V \to \mathbb{F}$.

(b) Assume that $f\colon V \times V \to \mathbb{F}$ is a non-degenerate alternating form. Prove that $\dim_{\mathbb{F}}(V)$ is even.

5. Let $E \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ be a subgroup of $GL(V)$ where V is a finite dimensional vector space over $\mathbb{R}$. Let $e_1, e_2, e_3 \in E$ denote the non-identity elements and, for $i = 1, 2, 3$, let V_i denote the eigenspace of e_i for the eigenvalue 1. Prove that $V = V_1 + V_2 + V_3$.

6. Let $V = \mathbb{R}^4$ and $\tau\colon V \to V$ have minimum polynomial $X^2 + X + 1$.

(a) Determine the minimum and characteristic polynomial of $\wedge^2(\tau)\colon \wedge^2(V) \to \wedge^2(V)$.

(b) Determine the minimum and characteristic polynomial of $\tau \otimes \tau\colon V \otimes V \to V \otimes V$.

7. Let U be the subgroup of $\mathbb{Z}^4$ generated by the elements $(-8, 8, 13, 2)$ and $(12, -12, -9, -2)$, and set $M := \mathbb{Z}^4/U$. Compute the invariant factors of M and the $\mathbb{Z}$-rank of $M/\mathrm{tor}(M)$.

8. Let $f(X) := X^6 - 3$ and let E be a splitting field of $f(X)$ over $\mathbb{Q}$.

(a) Compute $[E : \mathbb{Q}]$.

(b) Compute the Galois group of $E/\mathbb{Q}$ as a subgroup of S_6 by its action on the roots of $f(X)$.

(c) Compute the subextension F of $E/\mathbb{Q}$ with $[F : \mathbb{Q}] = 4$.

9. (a) Compute the cyclotomic polyonomial $\Phi_{12}(X)$.

(b) Let F be a splitting field of $\Phi_{12}(X)$ over $\mathbb{Q}$. Compute the Galois group of $F/\mathbb{Q}$.

(c) Find a Galois extension of $\mathbb{Q}$ with cyclic Galois group of order 12.