Algebra Qualifying Exam December 3, 1976

Group Theory

(1) (i) How many non-isomorphic abelian groups of order 1976 are there?

(ii) Use Sylow's theorem to show that any group of order 1976 is solvable.

(2) Let G be a group and $\mathcal{A} = \mathcal{A}(G) = \{\alpha : \alpha$ is an automorphism of $G\}$.

(i) Show that $\mathcal{A}$ is a group under the operation of composition.

(ii) Suppose G is a cyclic group. Show that $\mathcal{A}(G)$ is an abelian group.

(iii) If $g \in G$, let $\alpha_g \in \mathcal{A}(G)$ be defined by

$$\alpha_g(h) = g^{-1}hg \quad \text{for} \quad h \in G.$$

Show $\mathcal{I}(G) = \{\alpha_g : g \in G\}$ is a normal subgroup of $\mathcal{A}(G)$.

(iv) Show that there is a natural isomorphism $G/Z(G) \to \mathcal{I}(G)$ where $Z(G)$ is the center of G.

(3) Let G be a finite group of order n and p the smallest prime divisor of n. Suppose H is a subgroup of G and $|G: H| = p$. Prove $H \triangleleft G$.

Linear Algebra

(4) Let V be the set of all polynomials of degree $\leq n$ with real coefficients. Consider V as a vector space over the real numbers. Let

$D: V \to V$ denote the operation of differentiation.

(i) What is the dimension of V ?

(ii) Show D is a linear mapping of V into itself.

(iii) Find the dimensions of $\operatorname{Ker} D$, $\operatorname{Im} D$ and $\operatorname{Ker} D \cap \operatorname{Im} D$.

(iv) Let $E = D^2 + D + 1$. Give the characteristic polynomial of E and the eigenvalues of E.

(5) Let W be a finite dimensional vector space over a commutative field k. Let $\beta: W \times W \to k$ be a symmetric bilinear form on W. For any subspace $Y \subseteq W$ define $Y^\perp = \{w \in W: \beta(w,y) = 0$ for all $y \in Y\}$.

(i) Show that β induces a natural homomorphism $\hat{\beta}$ of W into its dual W^*.

(ii) Prove $\hat{\beta}$ is an isomorphism if and only if $W^\perp = 0$.

(iii) Prove that if $W^\perp = 0$, then $\dim Y + \dim Y^\perp = \dim W$.

Field Theory, Galois Theory

(6) Let $f \in Q[X]$ be an irreducible polynomial of degree 4 over the rationals. Suppose f has exactly two real roots. Prove that the Galois group of f is either the entire symmetric group S_4 or else has order 8.

(7) If p is a prime, then the ring $\mathbb{Z}/p\mathbb{Z} = \mathbb{F}p$ is a field with p elements.

(i) Show for each prime p and positive integer n, that there is a field with p^n elements. [Hint: Consider the splitting filed of the polynomial $x^{p^n} - x$ and shows its roots form a subfield].

(ii) Suppose K is a finite field and n a positive integer, prove that there is an irreducible polynomial of degree n over K.

Rings

(8) Suppose A is a commutative ring with identity and $A[X]$ is a PID. Prove A is a field.

(9) Consider the integral domain $\mathbb{Z}[\sqrt{m}] = \{a + b\sqrt{m}: a, b \in \mathbb{Z}\}$, m some integral non-square. Show that $\mathbb{Z}[\sqrt{m}]$ has a finite number of units if and only if $m < 0$.