Analysis Qualifying Examination
Winter Quarter 1982

January 14, 1982
9:00 am - 12:00 pm

PART I

Pick two out of three

1. Prove that a metric space in which every infinite set of points has a limit point is separable.

2. (a) State Fatou's lemma.

   (b) Using Lebesgue's Monotone Convergence theorem prove this lemma.

3. Suppose $\{A_n\}$ is a sequence of bounded linear transformations from a normed linear space X to a Banach space Y. Suppose $\|A_n\| \leq M < \infty$ for all n, and suppose there is a dense set $E \subset X$ such that $\{A_n x\}$ converges for each $x \in E$. Prove that $A_n x$ converges for each $x \in X$.

PART II

Pick two out of three

4. Show that $\|fgh\|_1 \leq \|f\|_p \, \|g\|_q \, \|h\|_r$;

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$$

where $\|f\|_p = \left(\int_a^b |f(x)|^p \, dx\right)^{1/p}$.

5. Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx$.

6. Let $\{x_n\}$ be a sequence such that the subsequence $\{x_{2n}\}$, $\{x_{2n+1}\}$ and x_{3n} converges. Does $\{x_n\}$ converge? Give a proof or counterexample.

PART III

Pick two out of three

7. Let $f_1(z),\ldots f_n(z)$ be holomorphic in a domain U . Suppose that $|f_1(z)|^2 + |f_2(z)|^2 + \ldots + |f_n(z)|^2$ is harmonic. Show that all the f_k are constant.

8. (a) Find a conformal map of the unit disc D onto the upper half plane.

   (b) Let $f: \overline{D} \to \mathbb{C}$ be holomorphic on D and continuous on $\overline{D}$. Suppose f is real on $\partial\overline{D}$. Show f is constant.

   Hint: Use part (a).

9. Show that if $|f(z)| \leq |p(z)|$ for sufficiently large z , p a polynomial of degree n , then f is a polynomial of degree n .