

Algebra Preliminary Exam

Fall 2008

1. Let $G = GL(n, \mathbb{C})$ be the group of all $n \times n$ invertible matrices with complex entries. Show that G contains a subgroup B with the following properties:

(a) B is *solvable*.

(b) $G = \cup_{g \in G} gBg^{-1}$, i.e. G is the union of the conjugates of B .

(Hint: consider the upper triangular matrices in G .)

2. Let G be a finite group of order pqr where p, q, r are *distinct primes*. Prove that G is *solvable*.

3. What is the automorphism group of $\mathbb{Z}_2 \times \mathbb{Z}_4$ isomorphic to? Justify your answer.

4. Let A be a $n \times n$ complex matrix such that A^k is the identity matrix for some $k > 0$. Prove that A is diagonalizable.

5. Let $A = J_6(\lambda)$ be a Jordan block of 6×6 matrix with eigenvalue λ .

(a) Suppose $\lambda = 0$. Find the Jordan canonical forms of A^2 and A^3 .

(b) Suppose $\lambda \neq 0$. Find the Jordan canonical forms for A^2 and A^3 .

6. Let A be a real symmetric and positive definite matrix. Prove that the maximal matrix entries are on the diagonal.

7. Let $I = (2, x)$ be the ideal generated by 2 and x in the ring $R = \mathbb{Z}[x]$. Show that the element $2 \otimes 2 + x \otimes x \in I \otimes_R I$ cannot be written as $a \otimes b$ for some $a, b \in I$.

8. (a) Show $(x^d - 1) \mid (x^n - 1)$ if and only if $d \mid n$.

(b) Let ℓ be a prime and let ζ_ℓ be a primitive ℓ th root of unity over \mathbb{F}_p for a prime p . Show that $\zeta_\ell \in \mathbb{F}_{p^n}$ if and only if $\ell \mid p^n - 1$.

(c) Find the splitting field of $x^7 - 1 \in \mathbb{F}_{11}[x]$.

9. Let $\alpha = \sqrt{2 + \sqrt{2}}$. Show that $\mathbb{Q}(\alpha)$ is a cyclic Galois extension of degree 4.