Extracting Order from Unpredictable Systems

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06/04/2018

Simple Equations

- Let's take a look at a simple function:

$$x_{n+1} = r\, x_n\, (1\text{-}x_n)$$

- This is a commonly used equation among scientists and biologists when trying to model populations.

- Even simple mathematical equations can produce complex behavior when there is feedback in the system.

- What is feedback, you may ask?

- Feedback is constant iterations of a recursive function

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- x_n is the current population
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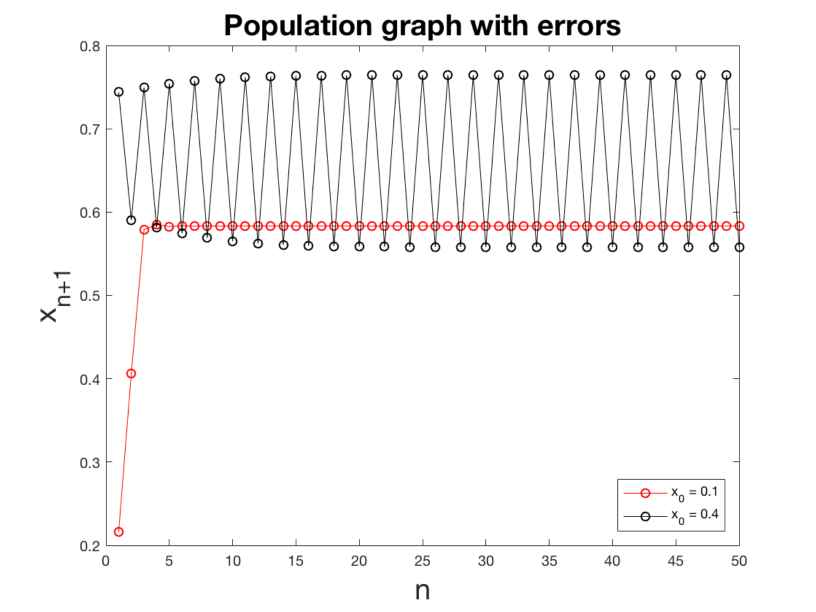
- $(1 - x_n)$ is added to the equation to keep the population within certain limits, so the numbers do not get out of hand

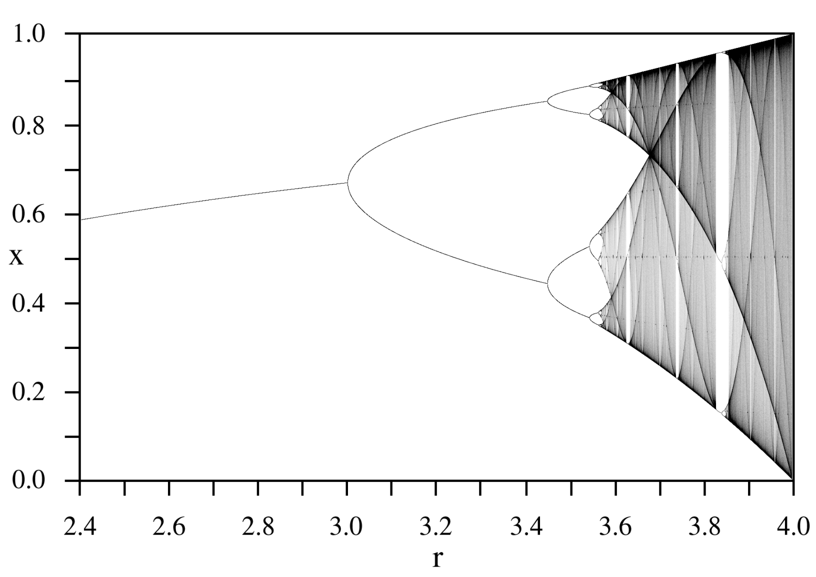
Simple Equations

- You can set x_0 to be a specific number, such as $x_0 = 0.1$ and pick a value for r (usually taken to be between 0 and 4), and see how the graph changes
- $r = 2.4$ and $r = 3.1$

n	$x_0 = 0.1$	$x_0 = 0.4$
1	0.2160	0.7440
2	0.4064	0.5904
3	0.5789	0.7496
4	0.5850	0.5818
5	0.5826	0.7542
6	0.5836	0.5745
7	0.5832	0.7577
8	0.5833	0.5690

Graphs


Population graph with errors
0.8
0.7
0.6
0.5
0.4
0.3
0.2
x_{n+1}
0
5
10
15
20
25
30
35
40
45
50
n
x_0 = 0.1
x_0 = 0.4


1.0
0.8
0.6
x
0.4
0.2
0.0
2.4
2.6
2.8
3.0
3.2
3.4
3.6
3.8
4.0
r

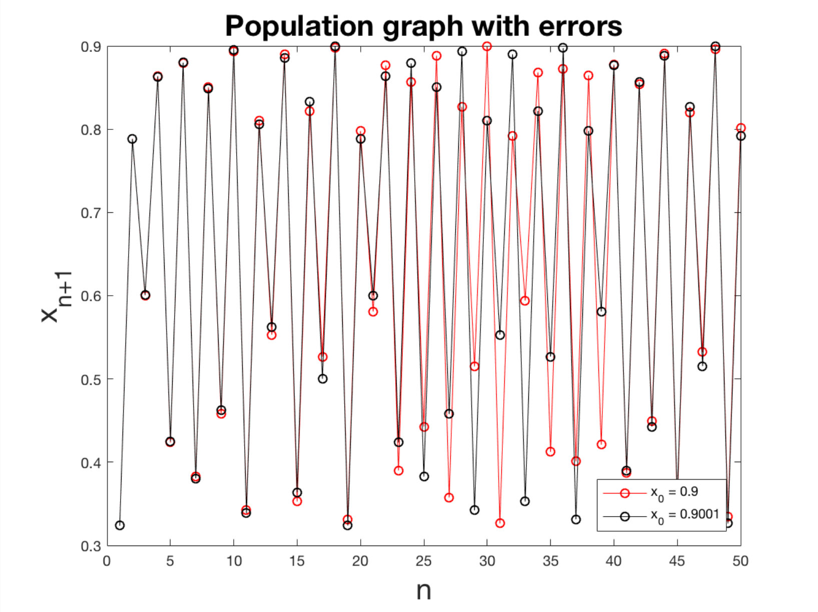
Bifurcation

- Even the simplest systems are capable of showing *both* orderly and chaotic behavior (Burbanks)
- Switching between order and chaos, (and also having multiple fixed points), is called a ”bifurcation.”

- The Bifurcation Map and Time vs Population graphs are related to each other because both involve r.
- The Bifurcation Map shows at the fixed points at different values of r.
- So if the graph splits into two, then for that value of x (which represents the population) you will have the population stabilize between two different values year after year.

- Now, what's so cool about modeling populations, you say?
- How easily little changes can make a big effect on them!
- Say we change the value of our x_0 from $x_0 = 0.9$ to $x_0 = 0.9001$ with $r = 3.6$
- The change is so small it can be considered a rounding error, but it can have quite a large effect on the graph.

Graph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x_0 = 0.9001

SIMPLE EQUATIONS

- As it turns out, our simple population function tends to amplify small errors until they become very large.
- So, after many iterations, the function will grow until our small error is as large as the x_0 values.
- No matter how small we make the error, it will eventually make a noticeable difference in most cases.
- This phenomenon is called Sensitive Dependence on Initial Conditions, which is when small errors tend to become amplified until they are as large as the other variables (Burbanks).

Hallmarks of Chaos

- Sensitive Dependence on Initial Conditions is one of the three hallmarks of the mathematical concept of chaos
- The other two are periodic orbits (think ”cycles” or repeating patterns) and ”mixing” behavior (think kneading dough) (Burbanks)

Sensitivity

- So, just how do we measure this sensitivity? After all, different errors will create differently amplified graphs.
- Enter Aleksandr M. Lyapunov—a Russian mathematician and physician who made very big contributions to the field of mathematics, namely the Lyapunov Exponent.
- The Lyapunov characteristic exponent of a dynamical system is a quantity that characterizes the rate of separation of infinitesimally close trajectories (Lyapunov exponent).

Lyapunov

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- Which we can write as $E_0 = x_0 + u_0$ and so the iterations will look like:

$$\begin{aligned} E_1 &= x_1 + u_1 \\ E_2 &= x_2 + u_2 \\ &\vdots \\ E_n &= x_n + u_n \end{aligned}$$

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- If we wanted to see how much the error is amplified after n steps, we can write it as:

$$|\frac{E_n}{E_0}| = |\frac{E_n}{E_{n-1}}| \times |\frac{E_{n-1}}{E_{n-2}}| \times \cdots \times |\frac{E_1}{E_0}|$$

Lyapunov

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- But we can expand $ln(\left|\frac{E_n}{E_0}\right|)$ to be:

$$ln(\left|\frac{E_n}{E_0}\right|) = ln(\left|\frac{E_n}{E_{n-1}}\right| \times \left|\frac{E_{n-1}}{E_{n-2}}\right| \times \cdots \times \left|\frac{E_1}{E_0}\right|)$$

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- As:

$$(\frac{1}{n})\sum_{k=1}^{k=n} ln(|\frac{E_n}{E_{n-1}}|)$$

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- If we substitute $h = E_{k-1}$ and $x = x_{k-1}$ then we get:

$$\frac{E_k}{E_{k-1}} = \frac{f(x+h)-f(x)}{h}$$

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- Hence:

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- And finally, if we let n go to ∞, we have:

$$(\frac{1}{n})\sum_{k=1}^{k=n} ln(|\frac{E_n}{E_{n-1}}|) = \lim_{n\to\infty}(\frac{1}{n})\sum_{k=1}^{k=n} ln(|f'(x_{n-1})|)$$

(Burbanks).

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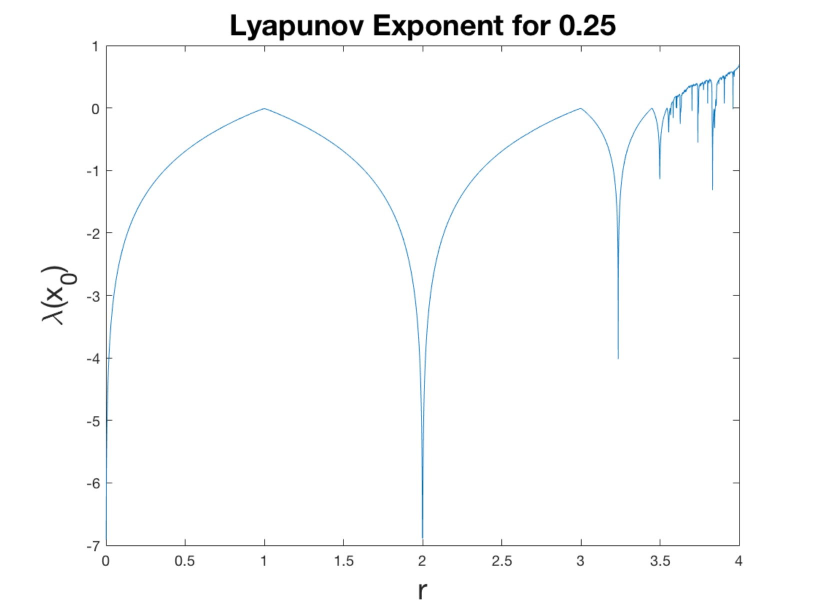
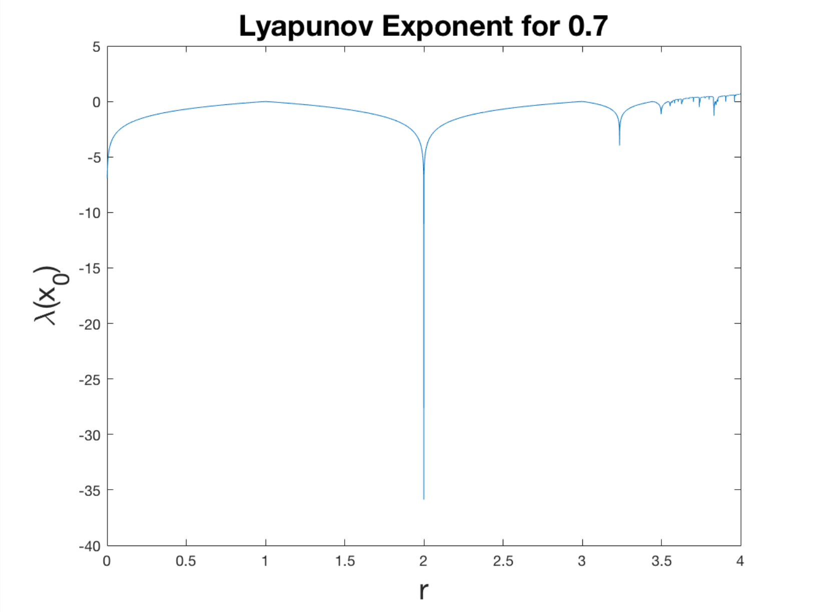

Lyapunov Exponent for 0.25
1
0
-1
-2
-3
-4
-5
-6
-7
$\lambda(x_0)$
0
0.5
1
1.5
2
2.5
3
3.5
4
r

Graphs



Lyapunov Exponent for 0.7
5
0
-5
-10
-15
-20
-25
-30
-35
-40
λ(x0)
0
0.5
1
1.5
2
2.5
3
3.5
4
r

Works Cited

Works Cited:

”Lyapunov exponent.” *Wikipedia, en.wikipedia.org/wiki/Lyapunov − exponent*. Accessed 3 June 2018.

Burbanks, Andy. ”Extracting beauty from chaos.” *+Plus Magazine, https : //plus.maths.org/content/extracting − beauty − chaos* Accessed 3 June 2018.