Analysis Preliminary Exam, Math @ UCSC, Spring 2020

1. Suppose that $\{f_n\}$ is a sequence of real-valued functions defined in $[0,1]$ that are continuous and monotonically increasing in $[0,1]$. And suppose that $\{f_n\}$ converges pointwisely to a function f in $[0,1]$. Assume that f is continuous in $[0,1]$. Show that $\{f_n\}$ actually converges to f uniformly in $[0,1]$.

2. Suppose that (X, d) is a bounded metric space. Let

$$d(x, A) := \inf\{d(x, a) : a \in A\}.$$

And define

$$d_H(A, B) := \inf\{\epsilon > 0 : A \subset N_\epsilon(B) \text{ and } B \subset N_\epsilon(A)\},$$

where

$$N_\epsilon(A) := \{x \in X : d(x, A) < \epsilon\}.$$

Show that d_H is a distance function on the space of all closed subsets in X.

3. Consider the measure space $(X, \mathcal{M}, \mu)$ with μ a positive measure. Suppose $\{E_k\}_{k\in\mathbb{N}} \subset \mathcal{M}$ satisfies $\sum_{k=1}^{\infty} \mu(E_k) < \infty$. Let

$$E := \{x \in \mathbb{R};\ x \in E_k \text{ for infinitely many } k\}.$$

   (a) Show that $E \in \mathcal{M}$.

   (b) Prove $\mu(E) = 0$.

4. Let $f\colon [0,1] \to \mathbb{R}$ be continuous, and let $g\colon [0,1] \to \mathbb{R}$ be measurable such that $0 \leq g(x) \leq 1$. Show that the limit

$$\lim_{n\to\infty} \int_0^1 f(g(x)^n)\, dx \qquad (dx : \text{ Lebesgue measure})$$

exists and compute it.

5. (a) Let X be a normed space and let Y be a Banach space. Show that the set $L(X, Y)$ of all bounded linear operators $A : X \to Y$ is a Banach space.

   (b) Show that the set G of all invertible (i.e., bijective) bounded linear operators on the Banach space Y is an open subset of $L(Y)$.

6. Let ℓ^∞ be the Banach space of all sequences $x = \{x_k\}_{k=1}^{\infty}$ with $\|x\|_\infty = \sup_{k\geq 1} |x_k| < \infty$. Show that there exists a bounded linear functional ϕ on ℓ^∞ with the property that

$$\phi(x) = \lim_{k\to\infty} x_k$$

whenever $x = \{x_k\}_{k=1}^{\infty}$ is a sequence for which the limit exists. If E denotes the set of all such functionals, describe the set $\{\ \|\phi\|\ :\ \phi \in E\}$.

7. Let $\Omega := \{z \in \mathbb{C} : |z| > 3\}$, $f(z) = \frac{z+1}{(z-2)(z^2+3)}$ and $g(z) = \frac{z^2}{(z-2)(z^2+3)}$. Is there a holomorphic function whose derivative is $f(z)$ on Ω? Is there a holomorphic function whose derivative is $g(z)$ on Ω? Justify your answers.

8. Suppose that $f(z)$ is an entire function and satisfies $|f(z)| \leq C(1+|z|)^{\frac{1}{2}}$ for all $z \in \mathbb{C}$ and some constant $C > 0$. Show that f is a constant.