Preliminary Examination in Algebra : Spring 2022

1. Let G be a finite group, p a prime, P a Sylow p-subgroup of G and U a subgroup of G containing $N_G(P)$. Show that $N_G(U) = U$.

2. Let G be a finite group. Show that G is solvable if and only if there exists a sequence $\{1\} = G_0 \leq G_1 \leq \cdots \leq G_r = G$ of normal subgroups G_i of G such that for each $i = 1, \ldots, r$, G_i/G_{i-1} is a group of prime power order.

3. Let p be an odd prime and $\mathbb{F}_p$ the field with p elements.

(a) Show that the congruence $x^2 \equiv -1 \mod p$ has a solution if and only if $p \equiv 1 \mod 4$.

(b) Determine the cardinality of $\mathrm{Hom}(\mathrm{Sym}(3), A)$, where A is the unit group of the ring $\mathbb{F}_p[X]/(X^2 + 1)$.

4. Let M_n denote the set of $n \times n$ complex matrices. For $A \in M_n$, we denote by A^* its conjugate transpose. Prove or disprove: For all $n \geq 1$ and $A \in M_n$, we have $A = A^*$ if and only if there exist a real number $r > 0$ and $B \in M_n$ such that

$$A = rI_n - B^*B,$$

where I_n denotes the identity matrix.

5. Let $V = \mathbb{F}_p^2$, $M = \mathrm{End}(V) = M_{2\times 2}(\mathbb{F}_p)$, and $G = GL(V) = GL_2(\mathbb{F}_p)$.

(a) Let G act on M by conjugation and let $\pi \in M$ denote the projection

$$\pi(x, y) = (x, 0) \quad \text{for all } x, y \in \mathbb{F}_p.$$

Compute the cardinality of (the stabiliser) $\mathrm{Stab}_G(\pi)$.

(b) Count the number of $A \in M$ such that $A^2 = A$.

6. Let $M = \mathbb{Z}^3$ and let N be the subgroup of M generated by the 3 elements

$$(2, 4, 6), \ (4, 5, 6), \ \text{and} \ (7, 8, 9).$$

Compute the rank and the elementary divisors of the quotient $\mathbb{Z}$-module M/N.

7. Describe a Galois extension of $\mathbb{Q}$ such that the Galois group is cyclic of order 7.

8. Let p be a prime and $\mathbb{F}_p$ be a a field with p elements. Find the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$ and justify your answer.

9. Determine the Galois group of $x^8 + 2 \in \mathbb{Q}[x]$.