Algebra Preliminary Exam, Fall 2020, UCSC

1. Show that the symmetric group Sym(7) has no subgroup of order 15.

2. Let $G = Q_8$ denote the quaternion group of order 8. Up to isomorphism, how many G-sets with 4 elements do there exist? (Justify your answer)

3. An element e of a ring R is called an *idempotent* if $e^2 = e$. Let e and f be idempotents of $Z(R)$ with $ef = 0$ and $e + f = 1$. Show that the ideals Re and Rf of R are again rings and that the rings R and $Re \times Rf$ are isomorphic.

4. Prove, or disprove with a counterexample, the following statement: Any 4×4 real matrix A that satisfies $A^3 + 4A = 5A^2$ is necessarily diagonalizable over the real numbers.

5. Let N be the subgroup of the free abelian group $M = \mathbb{Z}^3$ generated by the following 3 elements:

$$x = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \quad y = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad \text{and} \quad z = \begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix},$$

and let $Q := M/N$.

(a) Compute the rank of Q.

(b) Compute the invariant factors of Q.

6. Let p be a prime number and let $F = \mathbb{F}_p$ be the field with p elements. Count, with proof, the number of 3×3 matrices A with entries in F of rank 2 such that $A^2 = A$.

7. Find the Galois group of $x^6 - 4x^3 + 1$ over $\mathbb{Q}$.

8. Let p be an odd prime and $q = p^k$ with $k > 0$. Let E be a splitting field of $x^q + x \in \mathbb{F}_p[x]$. How many elements does E have? (Please give a proof of your answer)

9. Let L be a finite Galois extension of F. Let H_1 and H_2 be subgroups of $G = \mathrm{Gal}(L/F)$ and consider their fixed fields $E_1 = L^{H_1}$ and $E_2 = L^{H_2}$, respectively.

(a) Prove that $\mathrm{Gal}(L/E_1E_2) = H_1 \cap H_2$.

(b) Prove that $\mathrm{Gal}(L/E_1 \cap E_2)$ is the subgroup of G generated by H_1 and H_2.