Algebra Preliminary Exam, Fall 2021

1. Let $f\colon G \to H$ be a surjective group homomorphism. Show that $f(Z(G)) \subseteq Z(H)$ and give an example where $f(Z(G)) \neq Z(H)$.

2. Let G be a finite group, p a prime, and let $\mathcal{N}_p(G)$ denote the set of all normal subgroups N of G such that G/N is a p-group (i.e., $|G/N|$ is a power of p).

(a) Show that if $M, N \in \mathcal{N}_p(G)$ then $M \cap N \in \mathcal{N}_p(G)$.

(b) Set $O^p(G) := \bigcap_{N \in \mathcal{N}_p(G)} N$ and show that $O^p(G) \in \mathcal{N}_p(G)$.

(c) Let $f\colon G \to H$ be a group homomorphism. Show that there exists a group homomorphism $\bar{f}\colon G/O^p(G) \to H/O^p(H)$ such that $\bar{f}(gO^p(G)) = f(h)O^p(H)$.

3. Show that the polynomial

$$2021x^3 - \frac{3}{5}x^2 - \frac{4}{29}x + 7$$

is irreducible in $\mathbb{Q}[x]$.

4. Let A, B, and C be subspaces of a finite-dimensional vector space V. Suppose that $\dim(V) = n$. If $A \cap B = \{0\}$ and $B \cap C = \{0\}$ and $C \cap A = \{0\}$, prove that

$$\dim(A) + \dim(B) + \dim(C) \leq \frac{3}{2}n.$$

5. Consider the ring $M_2(\mathbb{F}_p)$ of two-by-two matrices with coefficients in the finite field with p elements. Define $\phi\colon M_2(\mathbb{F}_p) \to M_2(\mathbb{F}_p)$ by $\phi(X) = X^p$, the product of the matrix X with itself p times.

(a) Is ϕ an $\mathbb{F}_p$-linear map? Prove your answer.

(b) How many matrices $X \in M_2(\mathbb{F}_p)$ satisfy $\phi(X) = 0$? Justify your answer, taking care to consider the $p = 2$ case as well as the odd p case.

6. Let V be a finite-dimensional vector space over a field F of characteristic zero. Let $\phi\colon V \to V$ be a linear map. Let $[\phi \otimes \phi]\colon V \otimes V \to V \otimes V$ be the unique linear map satisfying $[\phi \otimes \phi](v_1 \otimes v_2) = \phi(v_1) \otimes \phi(v_2)$ for all $v_1, v_2 \in V$.

If $[\phi \otimes \phi]$ is the identity map, then what could ϕ be? Prove your answer. Hint: there are finitely many possibilities. Try the one and two dimensional cases first.

7. Let E be a field extension of the complex field $\mathbb{C}$ such that $[E : \mathbb{C}] = \dim_{\mathbb{C}} E$ is countable. Prove that $E = \mathbb{C}$.

8. Let E_1, E_2 be two finite extensions of a field F contained in a field E.

(a) Prove that $E_1 \otimes_F E_2$ is a field if and only if $[E_1E_2 : F] = [E_1 : F][E_2 : F]$.

(b) Prove that if both E_1, E_2 are Galois extensions of F then $E_1 \cap E_2$ is a Galois extension of F.

9. Determine whether or not $2x^5 - 10x + 5$ is solvable by radicals.